 Name: SOLUTION

Show all necessary work neatly, clearly, systematically, and understandably. Any incorrect statement and/or understatement may be penalized. There are 108 points available.

1. (8) Integrate: $\int \frac{x+1}{x^3-x^2}\,dx = \int \frac{x+1}{x^2(x-1)}\,dx$

PFD: $\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ $\times$ LCD $= x^2(x-1)$

$x+1 = Ax(x-1) + B(x-1) + Cx^2$

$x=0$: $1 = -B \Longrightarrow B = -1$

$x=1$: $2 = C$

x^2-term: $0 = A + C \Longrightarrow A = -C = -2$

So, $\int \frac{x+1}{x^3-x^2}\,dx = \int \left(-\frac{2}{x} + \frac{-1}{x^2} + \frac{2}{x-1}\right)dx$

$= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$

$= \frac{1}{x} + 2\ln\left|\frac{x-1}{x}\right| + C$

2. (10) Integrate: $\int \frac{2x^2+x-1}{x^3-3x^2+x-3}dx$. Note: YES, you need to know how to factor the denominator!

$$\frac{2x^2+x-1}{x^3-3x^2+x-3} = \frac{2x^2+x-1}{x^2(x-3)+(x-3)} = \frac{2x^2+x-1}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}$$

PFD : $2x^2+x-1 = A(x^2+1) + (x-3)(Bx+C)$

$x=3$: $2(3)^2+3-1 = 10A$

$20 = 10A$

$A = 2.$

$x=0$: $-1 = A - 3C$

$-1 = 2-3C \longrightarrow -3 = -3C$

$C = 1$

x^2-term : $2 = A + B$

$2 = 2 + B$

$B = 0$

So, $\int \frac{2x^2+x-1}{x^3-3x^2+x-3}dx = \int\left(\frac{2}{x-3} + \frac{1}{x^2+1}\right)dx$

$= 2\ln|x-3| + \tan^{-1}x + C$

3. (9) Integrate: $\int \frac{2x+1}{x^2-2x+2}\,dx$. Note: Complete the square.

$$= \int \frac{2x+1}{x^2-2x+1+1}\,dx$$

$$= \int \frac{2x+1}{(x-1)^2+1}\,dx$$

$$\begin{cases} u = x-1 \\ du = dx \\ u+1 = x \\ 2(u+1)+1 = 2x+1 \\ 2u+3 = 2x+1 \end{cases}$$

$$= \int \frac{2u-1}{u^2+1}\,du$$

$$= \int \frac{2u}{u^2+1}\,du - \int \frac{3}{u^2+1}\,du$$

$w = u^2+1$
$dw = 2u\,du$

$$= \int \frac{dw}{w} - 3\tan^{-1} u$$

$$= \ln|w| - 3\tan^{-1} u + C$$

$$= \ln|u^2+1| - 3\tan^{-1} u + C$$

$$= \ln|x^2-2x+1| - 3\tan^{-1}(x-1) + C$$

Alt

$$\int \frac{2x-2+3}{x^2-2x+2}\,dx$$

$$= \int \frac{2x-2}{x^2-2x+2}\,dx + \int \frac{3}{(x-1)^2+1}\,dx$$

$u = x^2-2x+2$ $\quad$ $w = x-1$
$du = (2x-2)\,dx$ $\quad$ $dw = dx$

$$= \int \frac{du}{u} + \int \frac{3}{w^2+1}\,dw$$

$$= \ln|u| + 3\tan^{-1} w + C$$

$$= \ln|x^2-2x+2| + 3\tan^{-1}(x-1) + C$$

4. (9) Integrate: $\displaystyle\int_{-1}^{8} \frac{dx}{\sqrt[3]{x}}$

$$= \int_{-1}^{0} x^{-\frac{1}{3}}\,dx + \int_{0}^{8} x^{-\frac{1}{3}}\,dx$$

$$= \lim_{a\to 0^-} \int_{-1}^{a} x^{-\frac{1}{3}}\,dx + \lim_{b\to 0^+} \int_{b}^{8} x^{-\frac{1}{3}}\,dx$$

$$= \lim_{a\to 0^-} \frac{3}{2} x^{\frac{2}{3}} \Big|_{-1}^{a} + \lim_{b\to 0^+} \frac{3}{2} x^{\frac{2}{3}} \Big|_{b}^{8}$$

$$= \lim_{a\to 0^-} \frac{3}{2} a^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}} + \frac{3}{2}(8)^{\frac{2}{3}} - \lim_{b\to 0^+} \frac{3}{2} x^{\frac{2}{3}}$$

$$= 0 + \frac{3}{2} + 6 - 0$$

$$= \frac{9}{2}$$

5. (9) Integrate: $\int_0^{\infty} x \cdot e^{-x}\, dx$

$$= \lim_{t\to\infty} \int_0^t x e^{-x}\, dx$$

$$\begin{array}{c|c} 1 & -e^{-x} \\ 0 & e^{-x} \end{array}$$

$$= \lim_{t\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left(-te^{-t} - e^{-t}\right) - (0-1)$$

$$= \lim_{t\to\infty} -te^{-t} - \lim_{t\to\infty} e^{-t} + 1$$

$$= \lim_{t\to\infty} -\frac{t}{e^t} - 0 + 1$$

$$\stackrel{LH}{=} \lim_{t\to\infty} \frac{-1}{e^t} + 1$$

$$= 0 + 1$$

$$= 1$$

6. (10:5,5) Let curve C: $y=\frac{2}{3}\left(x^2+1\right)^{3/2}$

a. Set up the integral that represents the length of curve C on $1 \le x \le 4$.

$$y' = \frac{2}{3}\cdot\frac{3}{2}\left(x^2+1\right)^{\frac{1}{2}}\cdot 2x = 2x\sqrt{x^2+1}$$

$$(y')^2 = 4x^2\left(x^2+1\right)$$

$$(y')^2+1 = 4x^4+4x^2+1 = \left(2x^2+1\right)^2$$

$$\sqrt{(y')^2+1} = \sqrt{\left(2x^2+1\right)^2} = 2x^2+1$$

$$\text{So, } L_{1,4} = \int_1^4 \left(2x^2+1\right)dx$$

b. Integrate the result from part (a). NOTE: No point will be given in this part if your set-up in part (a) is incorrect.

$$L_{1,4} = \int_1^4 \left(2x^2+1\right)dx$$

$$= \frac{2}{3}x^3+x \Big|_1^4$$

$$= \left(\frac{2}{3}\cdot 64+4\right)-\left(\frac{2}{3}+1\right)$$

$$= \left(\frac{128+12}{3}\right)-\left(\frac{5}{3}\right)$$

$$= \frac{135}{3}$$

$$= 45$$

7. (9:4,5) Let curve C: $y = x^3$.

a. Set up the integral that represents the surface area of <u>x-axis</u> revolution of curve C on $0 \le x \le 2$.

$y' = 3x^2$

$1 + (y')^2 = 1 + 9x^4$

$\sqrt{1+(y')^2} = \sqrt{1+(3x^2)^2} = \sqrt{1+9x^4}$

$S = \int_a^b 2\pi y \, ds.$

$S = 2\pi \int_0^2 x^3 \sqrt{1+9x^4} \, dx$

b. Integrate the result from part (a). NOTE: No point will be given in this part if your set-up in part (a) is incorrect.

$S = 2\pi \int_0^2 x^3 \sqrt{1+9x^4} \, dx$

$u = 1 + 9x^4$

$du = 36x^3 \, dx.$

$= 2\pi \cdot \frac{1}{36} \int_1^{145} u^{\frac{1}{2}} \, du$

$= \frac{\pi}{18} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{145}$

$= \frac{\pi}{27} \left(145^{\frac{3}{2}} - 1\right)$

$= \frac{\pi}{27} \left(145\sqrt{145} - 1\right)$

8. (11:5.6) Let curve C: $y = x^3$.

a. Set up the integral that represents the surface area of <u>y-axis</u> revolution of curve C on $0 \le x \le 2$.

$y' = 3x^2$

$1 + (y')^2 = 1 + (3x^2)^2$

$\sqrt{1+(y')^2} = \sqrt{1+(3x^2)^2}$

$S = \int_a^b 2\pi x \, ds$

$$S = 2\pi \int_0^2 x\sqrt{1+(3x^2)^2}\,dx$$

b. Integrate the result from part (a). NOTE: No point will be given in this part if your set-up in part (a) is incorrect.

$$S = 2\pi \int_0^2 x\sqrt{1+(3x^2)^2}\,dx$$

$u = 3x^2$

$du = 6x\,dx$

$= 2\pi \frac{1}{6}\int_0^{12} \sqrt{1+u^2}\,du$

$= \frac{\pi}{3}\int_0^{12}\sqrt{1+u^2}\,du$

$u = \tan\theta$

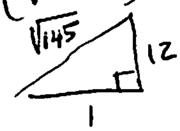
$du = \sec^2\theta\,d\theta$

$= \frac{\pi}{3}\int_0^{\tan^{-1}12} \sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta$

$= \frac{\pi}{3}\int_0^{\tan^{-1}12} \sec^3\theta\,d\theta$

$= \frac{\pi}{3}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]_0^{\tan^{-1}12}$

$\sec(\tan^{-1}12) = \sqrt{145}$



$= \frac{\pi}{6}\left(\sqrt{145}\cdot 12 + \ln\left|\sqrt{145}+12\right|\right)$

$= 2\pi\sqrt{145} + \frac{\pi}{6}\ln\left|\sqrt{145}+12\right|$

9. (11:4,3,3,1) Let lamina D be a region bounded by $\begin{cases} y = x^3 \\ y = \sqrt{x} \end{cases}$ with constant density ρ.

a. Find the mass of D.

$$m = \int_0^1 \rho(\sqrt{x} - x^3)\,dx$$
$$= \rho \int_0^1 (x^{\frac{1}{2}} - x^3)\,dx$$
$$= \rho\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^4\right)\Big|_0^1$$
$$= \rho\left(\frac{2}{3} - \frac{1}{4}\right)$$
$$= \frac{5}{12}\rho$$

$\rightarrow y = x^3 = \sqrt{x}$

$x^6 = x,\ x \geq 0$

$\Rightarrow x^6 - x = 0$

$x(x^5 - 1) = 0$

$x = 0,\ x = 1$

b. Find M_y, the moment about y-axis.

$$M_y = \int_0^1 \rho x(\sqrt{x} - x^3)\,dx$$
$$= \rho \int_0^1 (x^{\frac{3}{2}} - x^4)\,dx$$
$$= \rho\left(\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{5}x^5\right)\Big|_0^1$$
$$= \rho\left(\frac{2}{5} - \frac{1}{5}\right)$$
$$= \frac{1}{5}\rho$$

c. Find M_x, the moment about x-axis.

$$M_x = \frac{1}{2}\int_0^1 \rho\left(\sqrt{x}^2 - (x^3)^2\right)dx$$
$$= \frac{1}{2}\rho \int_0^1 (x - x^6)\,dx$$
$$= \frac{1}{2}\rho\left(\frac{1}{2}x^2 - \frac{1}{7}x^7\right)\Big|_0^1$$
$$= \frac{1}{2}\rho\left(\frac{1}{2} - \frac{1}{7}\right)$$
$$= \frac{1}{2}\rho \cdot \frac{5}{14}$$
$$= \frac{5}{28}\rho$$

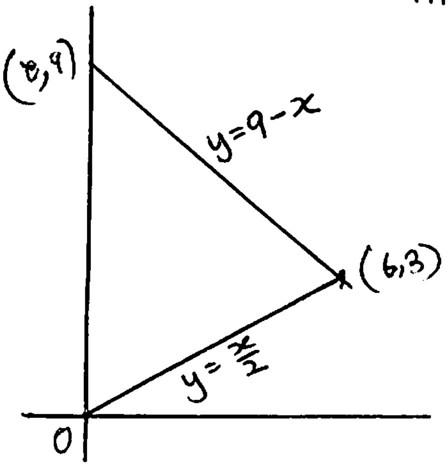
d. Find $\bar{y}$, the y-component of the centroid.

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{5}{28}\rho}{\frac{5}{12}\rho} = \frac{12}{28} = \frac{3}{7}$$

$\frac{3}{7}$

10. (12:4,4,4) Let lamina D be a triangle with vertices (0,0), (6,3) and (0,9) with constant density ρ.

a. Find the mass of D.



$$m = \rho\int_0^6 \left[(9-x) - \frac{x}{2}\right]dx$$
$$= \rho\int_0^6 \left(9 - \frac{3}{2}x\right)dx$$
$$= \rho\left(9x - \frac{3}{4}x^2\right)\Big|_0^6$$
$$= \rho(54 - 27)$$
$$= 27\rho$$

Simpler Method: $m = \rho \,\text{Area}$

$$m = \rho\, \frac{1}{2}\cdot 9\cdot 6$$
$$m = 27\rho.$$

b. Find M_x, the moment about x-axis.

$$M_x = \rho\int_0^6 \frac{1}{2}\left[(9-x)^2 - \left(\frac{x}{2}\right)^2\right]dx.$$
$$= \frac{\rho}{2}\int_0^6 \left(81 - 18x + x^2 - \frac{x^2}{4}\right)dx$$
$$= \frac{\rho}{2}\int_0^6 \left(81 - 18x + \frac{3}{4}x^2\right)dx$$
$$= \frac{\rho}{2}\left(81x - 9x^2 + \frac{1}{4}x^3\right)\Big|_0^6$$
$$= \frac{\rho}{2}\left(81\cdot 6 - 9\cdot 6^2 + \frac{1}{4}\cdot 6^3\right)$$
$$= \frac{\rho}{2}\cdot 6(81 - 54 + 9)$$
$$= 3\rho\cdot 36$$
$$= 108\rho.$$

c. Find M_x, the moment about x-axis.

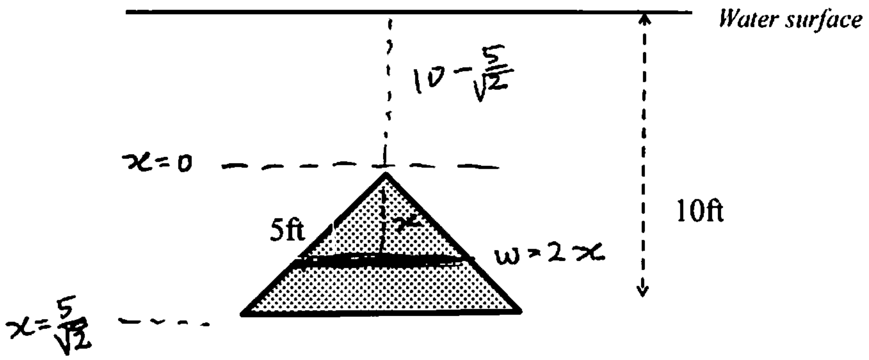
The question here should be:

$\bar{x} = ?$

$$\bar{x} = \frac{M_y}{m}$$
$$M_y = \rho\int_0^6 x\left[(9-x) - \frac{x}{2}\right]dx$$
$$= \rho\int_0^6 \left(9x - \frac{3}{2}x^2\right)dx$$
$$= \rho\left[\frac{9}{2}x^2 - \frac{1}{2}x^3\right]_0^6$$
$$= \rho\cdot 6^2\left(\frac{9}{2} - \frac{6}{2}\right).$$
$$= \rho\cdot 36\cdot\frac{3}{2}$$
$$= 54\rho.$$

So, $\bar{x} = \dfrac{54\rho}{27\rho} = 2.$

11. (10:5,5) A plate with the shape of <u>isosceles right</u> triangle of legs 5ft. each is submerged vertically into pool such that the hypotenuse is parallel to the water surface at 10 ft. depth.



a. Set up the integral that represents the hydrostatic force against the plate.

$$F = \int_0^{\frac{5}{\sqrt{2}}} \rho g\left(10-\frac{5}{\sqrt{2}}+x\right)2x\,dx.$$

$$= \rho g\int_0^{\frac{5}{\sqrt{2}}}\left(10-\frac{5}{\sqrt{2}}\right)2x\,dx + \rho g\int_0^{\frac{5}{\sqrt{2}}} 2x^2\,dx.$$

$$= \rho g\left(10-\frac{5}{\sqrt{2}}\right)x^2\Big|_0^{\frac{5}{\sqrt{2}}} + \rho g\frac{2}{3}x^3\Big|_0^{\frac{5}{\sqrt{2}}}$$

b. Integrate the result from part (a). NOTE: No point will be given in this part if your set-up in part (a) is incorrect.

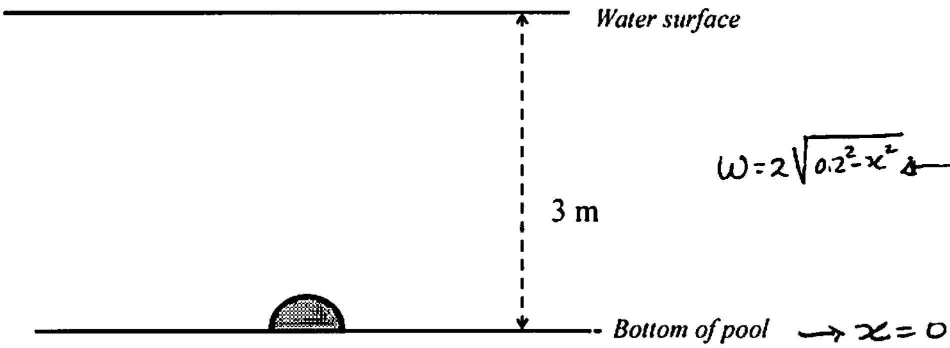
$$= \rho g\left(10-\frac{5}{\sqrt{2}}\right)\cdot\frac{25}{2} + \rho g\,\frac{2}{3}\cdot\frac{25}{2}\cdot\frac{5}{\sqrt{2}}$$

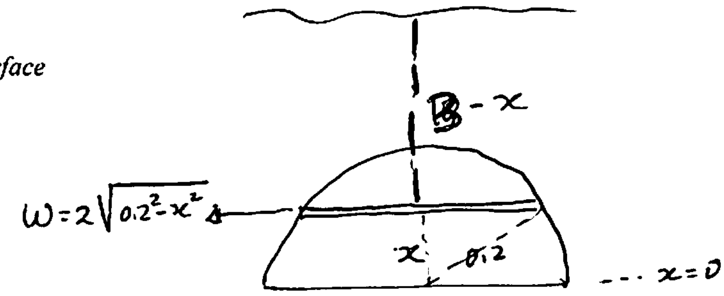
$$= \rho g\left[5\cdot 25 - \frac{125}{2\sqrt{2}} + \frac{125}{3\sqrt{2}}\right]$$

$$= 125\rho g\left(1-\frac{1}{6\sqrt{2}}\right).$$

$$= 125\rho g\left(\frac{12-\sqrt{2}}{12}\right).$$

12. (10:5,5) A drain with the shape of a semicircle with radius 0.2 m is located at the bottom wall of a pool with 3 m. depth.





a. Set up the integral that represents the hydrostatic force against the cover of that drain.

$$F = \int_0^{0.2} \rho g (3 - x)\, 2\sqrt{0.2^2 - x^2}\, dx$$

$$= \rho g \Big[6 \underbrace{\int_0^{0.2} \sqrt{0.04 - x^2}\, dx}_{\text{Area of } \frac{1}{4} \text{ circle with } r = 0.2} - \int_0^{0.2} 2x\sqrt{0.04 - x^2}\, dx \Big]$$

b. Integrate the result from part (a). NOTE: No point will be given in this part if your set-up in part (a) is incorrect.

$$= \rho g \Big[6 \cdot \frac{1}{4} \cdot \pi \cdot 0.2^2 - \int_0^{0.2} 2x \sqrt{0.04 - x^2}\, dx \Big]$$

$u = 0.04 - x^2$
$du = -2x\,dx$

$$= \rho g \Big[0.06\pi + \int_{0.04}^{0} u^{\frac{1}{2}}\, du \Big]$$

$$= \rho g \Big(0.06\pi - \frac{2}{3} u^{3/2} \Big|_0^{0.04} \Big)$$

$$= \rho g \Big(0.06\pi - \frac{2}{3} (0.2)^3 \Big)$$

$$= \rho g \Big(0.06\pi - \frac{0.016}{3} \Big)$$