Spring 20 Math 262 Test 3 Name: SOLUTION

<u>Show all necessary work neatly, clearly, systematically, and understandably. Any incorrect statement and/or understatement may be penalized. There are 104 points available.</u>

1. (24:5,6,4,6,3) Given the curve $\begin{cases} x = t^3 - t \longrightarrow x' = 3t^2 - 1 \\ y = t^2 \longrightarrow y' = 2t \end{cases}$

a. Find the coordinates of any horizontal and vertical tangents.

HT: $\frac{dy}{dx} = 0 \Longrightarrow y' = 0$ when $x' \neq 0$

$2t = 0$

$t = 0 \longrightarrow x'(0) = -1 \neq 0$

$(x,y)_{t=0} = (0,0)$.

VT: $\frac{dx}{dy} = 0 \Rightarrow x' = 0$ when $y' \neq 0$

$3t^2 - 1 = 0$

$t = \pm\frac{1}{\sqrt{3}} \rightarrow y'(\pm\frac{1}{\sqrt{3}}) = \pm\frac{2}{\sqrt{3}} \neq 0$

$(x,y)_{t=-\frac{1}{\sqrt{3}}} = \left(-\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}}, \frac{1}{3}\right) = \left(\frac{2\sqrt{3}}{9}, \frac{1}{3}\right)$

$(x,y)_{t=\frac{1}{\sqrt{3}}} = \left(\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}}, \frac{1}{3}\right) = \left(-\frac{2\sqrt{3}}{9}, \frac{1}{3}\right)$

b. Find $\frac{d^2y}{dx^2}$ and determine for which values of t the curve is concave down.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{2t}{3t^2-1}\right)}{3t^2-1} = \frac{2(3t^2-1) - 2t(6t)}{(3t^2-1)^3}$$

$$= \frac{6t^2 - 2 - 12t^2}{(3t^2-1)^3} = \frac{-6t^2-2}{(3t^2-1)^3} = -\frac{2(3t^2+1)}{(3t^2-1)^3} \overset{?}{<} 0$$

1) <u>C#</u>: $3t^2 - 1 = 0$

$1 = \pm\frac{1}{\sqrt{3}}$

2) <u>SG</u> $\ominus$ UD $\oplus$ UD $\ominus$; $-\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$

3) So, concave down on $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$

Continue fro #1: Given the curve $\begin{cases} x = t^3 - t \\ y = t^2 \end{cases}$

c. Find the x- and y- intercepts, if any.

x-int : $y = 0$

$t^2 = 0$

$t = 0$

$(x,y)_{t=0} = (0,0)$

y-int : $x = 0$

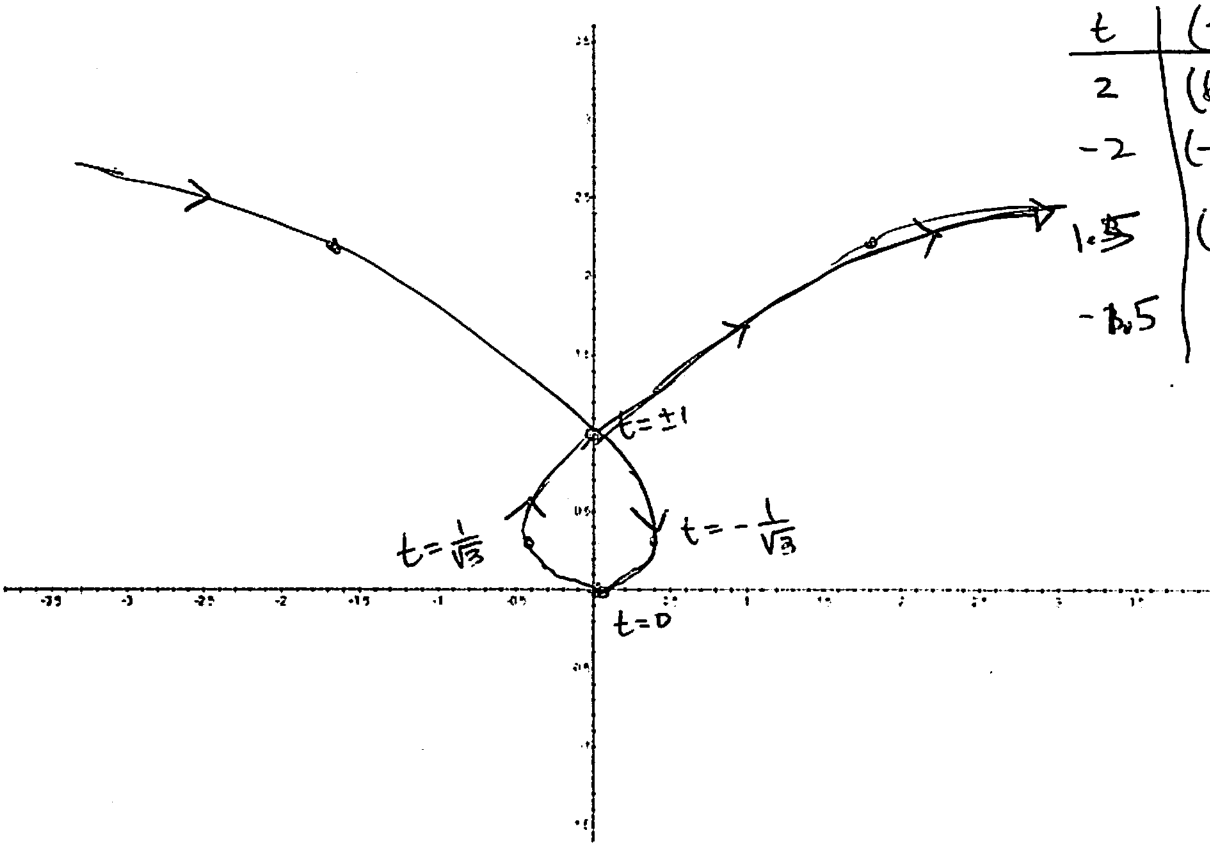
$t^3 - t = 0$

$t(t^2 - 1) = 0$

$t = 0, \ t = \pm 1$

$(x,y)_{t=\pm 1} = (0,1), \ (x,y)_{t=0} = (0,0)$

d. Sketch the graph of the curve using the information obtained earlier. Include direction of motion.



t	(x,y)
2	(6, 4)
-2	(-6, 4)
1.5	(1.875, 2.25)
-1.5	

e. Find the equation(s) of tangent line(s) at t = -2. $(x,y)\big|_{t=2} = (-6,4)$.

$$\left.\frac{dy}{dx}\right|_{t=-2} = \left.\frac{2t}{3t^2-1}\right|_{t=-2} = \frac{-4}{12-1} = -\frac{4}{11}$$

$\Rightarrow$ TL : $y = -\frac{4}{11}x + b$

$4 = -\frac{4}{11}(-6) + b$

$4 = \frac{24}{11} + b$

$\frac{20}{11} = b$

$y = -\frac{4}{11}x + \frac{20}{11}$

$4x + 11y = 20$

2. (18:4,6,8) Let curve C: $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$, $0 \le t \le 2\pi$ $\Rightarrow \begin{cases} x' = 1 - \cos t \\ y' = \sin t \end{cases}$

a. Find the area under C bounded by the x-axis.

$$A = \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt$$

$$= 2\int_0^{\frac{\pi}{2}} (1-\cos t)^2\,dt$$

$$= 2\int_0^{\frac{\pi}{2}} (1 - 2\cos t + \cos^2 t)\,dt$$

$$= 4\int_0^{\frac{\pi}{2}} \left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt$$

$$= 4\int_0^{\frac{\pi}{2}} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right)dt$$

$$= 4\left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 6\left(\frac{\pi}{2}\right)$$

$$= 3\pi$$

b. Find the arclength of C.

$$(x')^2 + (y')^2 = (1-\cos t)^2 + \sin^2 t$$

$$= 1 - 2\cos t + \cos^2 t + \sin^2 t$$

$$= 2 - 2\cos t = 4\left(\frac{1-\cos t}{2}\right)$$

$$= 4\sin^2\left(\frac{t}{2}\right).$$

$$\sqrt{(x')^2 + (y')^2} = 2\sin\left(\frac{t}{2}\right).$$

$$L = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right)dt$$

$$= -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi}$$

$$= +4 + 4$$

$$= 8.$$

c. Find the area of the surface generated by revolving about the x-axis.

$$SA = 2\pi\int_a^b y\,ds$$

$$= 2\pi\int_0^{2\pi} (1-\cos t)\, 2\sin\left(\frac{t}{2}\right)dt$$

$$= 2\pi\int_0^{2\pi} 2\sin\left(\frac{t}{2}\right)dt - 2\pi\cdot 2\int_0^{2\pi} \cos t \sin\left(\frac{t}{2}\right)dt$$

$$= 2\pi\cdot 8 - 4\pi\int_0^{2\pi} \frac{1}{2}\left[\sin\left(\frac{3t}{4}\right) - \sin\left(\frac{t}{4}\right)\right]dt$$

$$SA = 16\pi - 2\pi\left[-\frac{4}{3}\cos\left(\frac{3t}{4}\right) + 4\cos\left(\frac{t}{4}\right)\right]_0^{2\pi}$$

$$= 16\pi - 2\pi\left[\cancel{0} + \frac{4}{3} + 0 - 4\right]$$

$$= 16\pi - 2\pi\left(-\frac{8}{3}\right).$$

$$= 16\pi + \frac{16\pi}{3} = \frac{64\pi}{3}$$

3. (28:8,6,7,7) Let $C: r = 3 - 3\cos\theta$

a. Find the points at which the curve has horizontal and vertical tangent line.

HTL: $\frac{dy}{dx} = 0$

VTL: $\frac{dx}{dy} = 0$

So,

HTL: $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$(x,y)_{\theta=\frac{2\pi}{3}} = (-2.25, 2.25\sqrt{3})$

$(x,y)_{\theta=\frac{4\pi}{3}} = (-2.25, -2.25\sqrt{3})$

VTL: $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$(x,y)|_{\theta=\frac{\pi}{3}} = (0.75, 0.75\sqrt{3})$

$(x,y)|_{\theta=\frac{5\pi}{3}} = (0.75, -0.75\sqrt{3})$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 3\sin^2\theta + 3(1-\cos\theta)\cos\theta$

$= 3\sin^2\theta + 3\cos\theta - 3\cos^2\theta \stackrel{.}{=} 0$

$3\cos\theta = 3\cos 2\theta$

$\cos\theta = \cos 2\theta$

$0 = 2\cos^2\theta - \cos\theta - 1$

$0 = (2\cos\theta + 1)(\cos\theta - 1)$

$\cos\theta = -\frac{1}{2}, \cos\theta = 1$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

$= 3\sin\theta\cos\theta - 3(1-\cos\theta)\sin\theta$

$= 6\sin\theta\cos\theta - 3\sin\theta \stackrel{.}{=} 0$

$3\sin\theta(2\cos\theta - 1) \stackrel{.}{=} 0$

$\sin\theta = 0 \quad \cos\theta = \frac{1}{2}$

$\theta \neq 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$

At $\theta = 0$: $\lim_{\theta\to 0} \frac{3(\cos\theta - \cos 2\theta)}{3\sin\theta(2\cos\theta - 1)}$

$= \lim_{\theta\to 0} \frac{(1 + \cos\theta - 2\cos^2\theta)}{\sin\theta(2\cos\theta - 1)}$

$= \lim_{\theta\to 0} \frac{1 + \cos\theta(-1 - 2\cos\theta)}{\sin\theta(2\cos\theta - 1)}$

$= \lim_{\theta\to 0} -\frac{(\cos\theta - 1)}{\sin\theta}\frac{2\cos\theta + 1}{2\cos\theta - 1}$

$= \lim_{\theta\to 0} -\frac{\sin\theta}{\cos\theta} \lim_{\theta\to 0}\frac{2\cos\theta + 1}{2\cos\theta - 1}$

$= \underbrace{0 \cdot 3} = 0$

So, HTL at $\theta = 0$.

$(x,y)_{\theta=0} = (0,0)$.

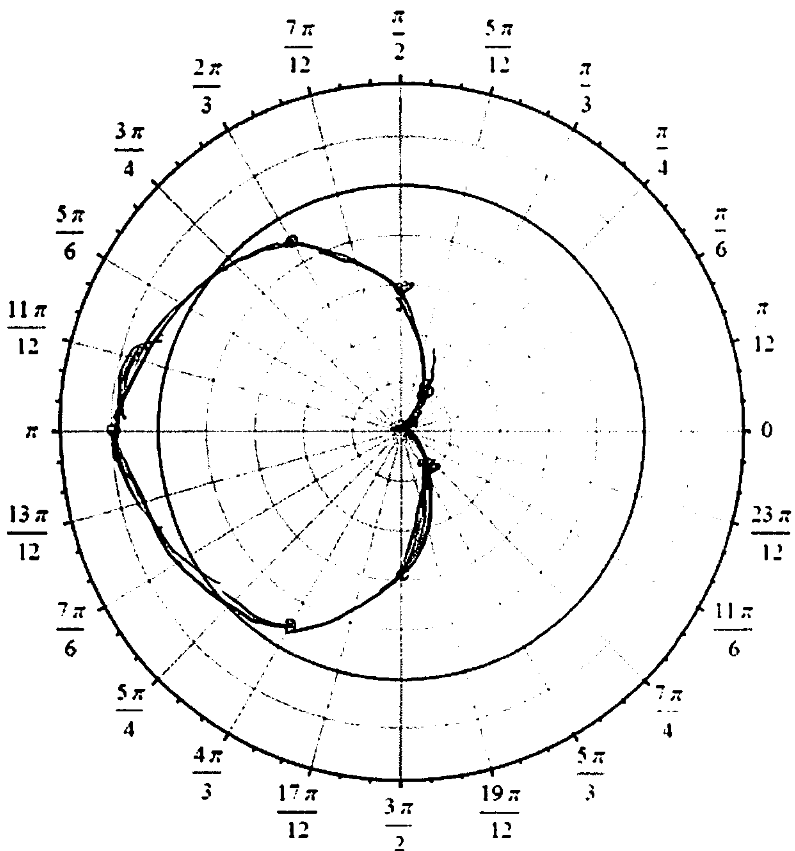
HTL at polar:

$(0,0)_p, (4.5, \frac{2\pi}{3})_p, (4.5, \frac{4\pi}{3})_p$

VTL at polar

$(1.5, \frac{\pi}{3})_p, (1.5, \frac{5\pi}{3})_p, (6, \pi)_p$.

b. Sketch C



Continue from #3: Let $C: r = 3 - 3\cos\theta$

c. Find the area of the region bounded by C

$$\text{Area} = 2 \cdot \frac{1}{2} \int_0^{\pi} (3-3\cos\theta)^2 \, d\theta$$

$$= 9 \int_0^{\pi} (1-\cos\theta)^2 \, d\theta$$

$$= 9 \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= 9 \int_0^{\pi} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 9 \int_0^{\pi} \left(\frac{3}{2} - 2\cancelto{0}{\cos\theta} + \frac{1}{2}\cancelto{0}{\cos 2\theta}\right) d\theta$$

$$= 9 \cdot \frac{3}{2} \pi$$

$$= \frac{27}{2} \pi$$

d. Find the perimeter of C.

$$\frac{dr}{d\theta} = 3\sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (3 - 3\cos\theta)^2 + (3\sin\theta)^2$$

$$= 9(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)$$

$$= 9 \cdot 2 (1 - \cos\theta)$$

$$= 18 \cdot 2 \sin^2\left(\frac{\theta}{2}\right)$$

$$= 36 \sin^2\left(\frac{\theta}{2}\right)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 6 \sin\left(\frac{\theta}{2}\right)$$

$$P = 2 \int_0^{\pi} 6 \sin\left(\frac{\theta}{2}\right) d\theta$$

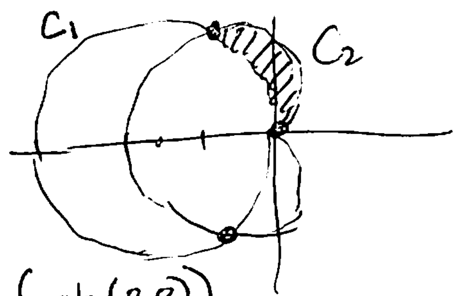
$$= 12 \left(-2\cos\left(\frac{\theta}{2}\right)\right)\Big|_0^{\pi}$$

$$= 24$$

4. (16:6.10) Let curves $\begin{cases} C_1 : r = -3\cos\theta \\ C_2 : r = 1 - \cos\theta \end{cases}$.

a. Find the angle θ where the curves intersect. Hint: It will be helpful if you sketch the graph

$$-3\cos\theta = 1 - \cos\theta$$
$$-2\cos\theta = 1$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$



Also at $\theta_2 = 0$ when $\theta_1 = \frac{\pi}{2}$ (at $(0,0)$).

b. Find the area of the region inside C_2 and outside C_1.

We will use symmetry

$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\frac{2\pi}{3}} (1-\cos\theta)^2\,d\theta - \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-3\cos\theta)^2\,d\theta\right]$$

$$= \int_0^{\frac{2\pi}{3}} (1-\cos\theta)^2\,d\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-3\cos\theta)^2\,d\theta$$

$$= \int_0^{\frac{2\pi}{3}} (1 - 2\cos\theta + \cos^2\theta)\,d\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 9\cos^2\theta\,d\theta$$

$$= \int_0^{\frac{2\pi}{3}} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta - \frac{9}{2}\int_{\frac{\pi}{2}}^{2\pi/3} (1+\cos 2\theta)\,d\theta$$

$$= \frac{3}{2}\left(\frac{2\pi}{3}\right) - 2\sin\theta\Big|_0^{\frac{2\pi}{3}} + \frac{1}{4}\sin 2\theta\Big|_0^{\frac{2\pi}{3}} - \frac{9}{2}\left(\frac{\pi}{6}\right) - \frac{9}{4}\sin 2\theta\Big|_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$$

$$= \pi - \sqrt{3} - \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{9\sqrt{3}}{8} + 0$$

$$= \frac{\pi}{4} - \frac{\sqrt{3}}{8}$$

$$\cong 0.56889$$

5. (13:4,5,4) Determine whether the sequence converges or diverges, and if it converges, find the limit. Use correct notation, format, explanation, and state any theorems used.

a. $a_n = \dfrac{\sin n}{n}$

$|a_n| = \left|\dfrac{\sin n}{n}\right| \le \dfrac{1}{n} \longrightarrow 0$

$\Longrightarrow |a_n| \to 0$ by squeeze Thm

Since $|a_n| \to 0$

then $a_n \to 0$.

c. $a_n = \dfrac{n^n}{n!} = \dfrac{n}{n} \cdot \dfrac{n}{n-1} \cdot \dfrac{n}{n-2} \cdot \dfrac{n}{n-3} \cdots \dfrac{n}{1}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$1 \quad >1 \quad >1 \quad >1 \cdots >1 \quad n$

$\geq n \longrightarrow \infty$

So, $a_n \to \infty$ by comparison

b. $a_n = n \tan\left(\dfrac{1}{n}\right)$

Let $f(x) = x \tan\left(\frac{1}{x}\right)$

note that $f(n) = a_n$.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{\sec^2\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$

$t = \frac{1}{x}$

$= \lim\limits_{t\to 0^+} \dfrac{\tan t}{t}$

$= 1$

$= \lim\limits_{x\to\infty} \sec^2\left(\frac{1}{x}\right)$

$= \sec^2 0$

$= 1$

By CSF, $a_n \to 1$.

6. (7) Find the limit $a_n = \left(1+\frac{b}{n}\right)^n$

Let $f(x) = \left(1+\frac{b}{x}\right)^x$, note $f(n) = a_n$.

Let $y = \left(1+\frac{b}{x}\right)^x$

$$\ln y = x \ln\left(1+\frac{b}{x}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1+\frac{b}{x}\right).$$

$$\ln\left(\lim_{x\to\infty} y\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{b}{x}\right)}{\frac{1}{x}}$$

$$\stackrel{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{b}{x}}\cdot\left(-\frac{b}{x^2}\right)}{-\frac{1}{x^2}}.$$

$$= \lim_{x\to\infty} \frac{b}{1+\frac{b}{x}}$$

$$= \frac{b}{1+0} = b$$

$$\lim_{x\to\infty} y = e^b$$

So, $a_n \longrightarrow e^b$