

Show all necessary work neatly, clearly, systematically, and understandably. Any incorrect statement and/or understatement may be penalized. There are 104 points available.

1. (24:5,6,4,6,3) Given the curve $\begin{cases} x = t^3 - t & \longrightarrow x' = 3t^2 - 1 \\ y = t^2 & \longrightarrow y' = 2t \end{cases}$

a. Find the coordinates of any horizontal and vertical tangents.

HT: $\frac{dy}{dx} = 0 \implies y' = 0$ when $x' \neq 0$
 $2t = 0$
 $t = 0 \longrightarrow x'(0) = -1 \neq 0$

$(x, y)_{t=0} = (0, 0)$

VT: $\frac{dx}{dy} = 0 \implies x' = 0$ when $y' \neq 0$
 $3t^2 - 1 = 0$

$t = \pm \frac{1}{\sqrt{3}} \longrightarrow y'(\pm \frac{1}{\sqrt{3}}) = \pm \frac{2}{\sqrt{3}} \neq 0$

$(x, y)_{t = -\frac{1}{\sqrt{3}}} = (-\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}}, \frac{1}{3}) = (\frac{2\sqrt{3}}{9}, \frac{1}{3})$

$(x, y)_{t = \frac{1}{\sqrt{3}}} = (\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}}, \frac{1}{3}) = (-\frac{2\sqrt{3}}{9}, \frac{1}{3})$

b. Find $\frac{d^2y}{dx^2}$ and determine for which values of t the curve is concave down.

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} = \frac{d \left(\frac{2t}{3t^2-1} \right)}{3t^2-1} = \frac{2(3t^2-1) - 2t(6t)}{(3t^2-1)^3}$$

$$= \frac{6t^2 - 2 - 12t^2}{(3t^2-1)^3} = \frac{-6t^2 - 2}{(3t^2-1)^3} = -\frac{2(3t^2+1)}{(3t^2-1)^3} < 0$$

1) C# : $3t^2 - 1 = 0$
 $t = \pm \frac{1}{\sqrt{3}}$

2) SG $\begin{array}{c} \ominus \quad \cup D \quad \oplus \quad \cup D \quad \ominus \\ \hline -\frac{1}{\sqrt{3}} \qquad \qquad \frac{1}{\sqrt{3}} \end{array}$

3) So, concave down on $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$

Continue from #1: Given the curve $\begin{cases} x = t^3 - t \\ y = t^2 \end{cases}$

c. Find the x- and y- intercepts, if any.

x-int: $y = 0$
 $t^2 = 0$
 $t = 0$

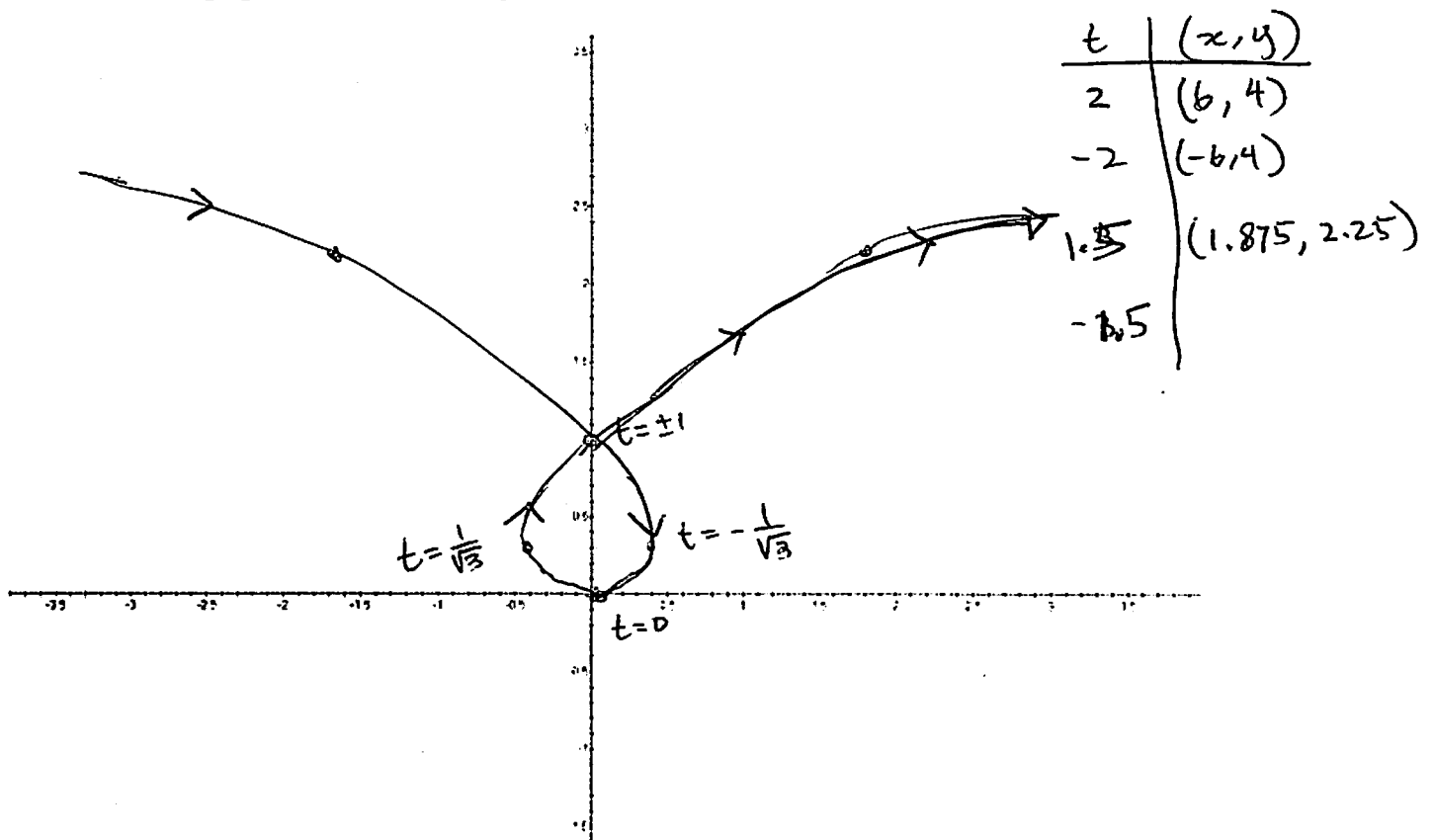
$(x, y)_{t=0} = (0, 0)$

y-int: $x = 0$
 $t^3 - t = 0$

$t(t^2 - 1) = 0$
 $t = 0, t = \pm 1$

$(x, y)_{t=\pm 1} = (0, 1), (x, y)_{t=0} = (0, 0)$

d. Sketch the graph of the curve using the information obtained earlier. Include direction of motion.



e. Find the equation(s) of tangent line(s) at $t = -2$.

$(x, y)_{t=-2} = (-6, 4)$

$\frac{dy}{dx} \Big|_{t=-2} = \frac{2t}{3t^2 - 1} \Big|_{t=-2} = \frac{-4}{12 - 1} = -\frac{4}{11}$

\Rightarrow TL: $y = -\frac{4}{11}x + b$
 $4 = -\frac{4}{11}(-6) + b$
 $4 = \frac{24}{11} + b$
 $\frac{20}{11} = b$

$y = -\frac{4}{11}x + \frac{20}{11}$

 $4x + 11y = 20$

2. (18.4.6,8) Let curve C: $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}, 0 \leq t \leq 2\pi \Rightarrow \begin{cases} x' = 1 - \cos t \\ y' = \sin t \end{cases}$

a. Find the area under C bounded by the x-axis.

$$\begin{aligned} A &= \int_0^{2\pi} (1 - \cos t)(1 - \cos t) dt \\ &= \int_0^{\frac{\pi}{2}} (1 - \cos t)^2 dt \\ &= \int_0^{\frac{\pi}{2}} (1 - 2\cos t + \cos^2 t) dt \\ &= \int_0^{\frac{\pi}{2}} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right) dt \\ &= \left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right) \Big|_0^{\frac{\pi}{2}} \\ &= 6\left(\frac{\pi}{2}\right) \\ &= 3\pi \end{aligned}$$

b. Find the arclength of C.

$$\begin{aligned} (x')^2 + (y')^2 &= (1 - \cos t)^2 + \sin^2 t \\ &= 1 - 2\cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2\cos t = 4\left(\frac{1 - \cos t}{2}\right) \\ &= 4\sin^2\left(\frac{t}{2}\right) \end{aligned}$$

$$\sqrt{(x')^2 + (y')^2} = 2\sin\left(\frac{t}{2}\right)$$

$$\begin{aligned} L &= \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt \\ &= -4\cos\left(\frac{t}{2}\right) \Big|_0^{2\pi} \\ &= +4 + 4 \\ &= 8 \end{aligned}$$

c. Find the area of the surface generated by revolving about the x-axis.

$$\begin{aligned} SA &= 2\pi \int_a^b y ds \\ &= 2\pi \int_0^{2\pi} (1 - \cos t) 2\sin\left(\frac{t}{2}\right) dt \\ &= 2\pi \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt \\ &\quad - 2\pi \cdot 2 \int_0^{2\pi} \cos t \sin\left(\frac{t}{2}\right) dt \\ &= 2\pi \cdot 8 - 4\pi \int_0^{2\pi} \frac{1}{2} \left[\sin\left(\frac{3t}{4}\right) - \sin\left(\frac{t}{4}\right) \right] dt \\ SA &= 16\pi - 2\pi \left[-\frac{4}{3} \cos\left(\frac{3t}{4}\right) + 4\cos\left(\frac{t}{4}\right) \right]_0^{2\pi} \\ &= 16\pi - 2\pi \left[\frac{4}{3} + \frac{4}{3} + 0 - 4 \right] \\ &= 16\pi - 2\pi \left(-\frac{8}{3} \right) \\ &= 16\pi + \frac{16\pi}{3} = \frac{64\pi}{3} \end{aligned}$$

3. (28:8,6,7,7) Let $C: r = 3 - 3\cos\theta$

a. Find the points at which the curve has horizontal and vertical tangent line.

HTL: $\frac{dy}{dx} = 0$

VTL: $\frac{dx}{dy} = 0$

So,

HTL: $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$(x, y)_{\theta = \frac{2\pi}{3}} = (-2.25, 2.25\sqrt{3})$

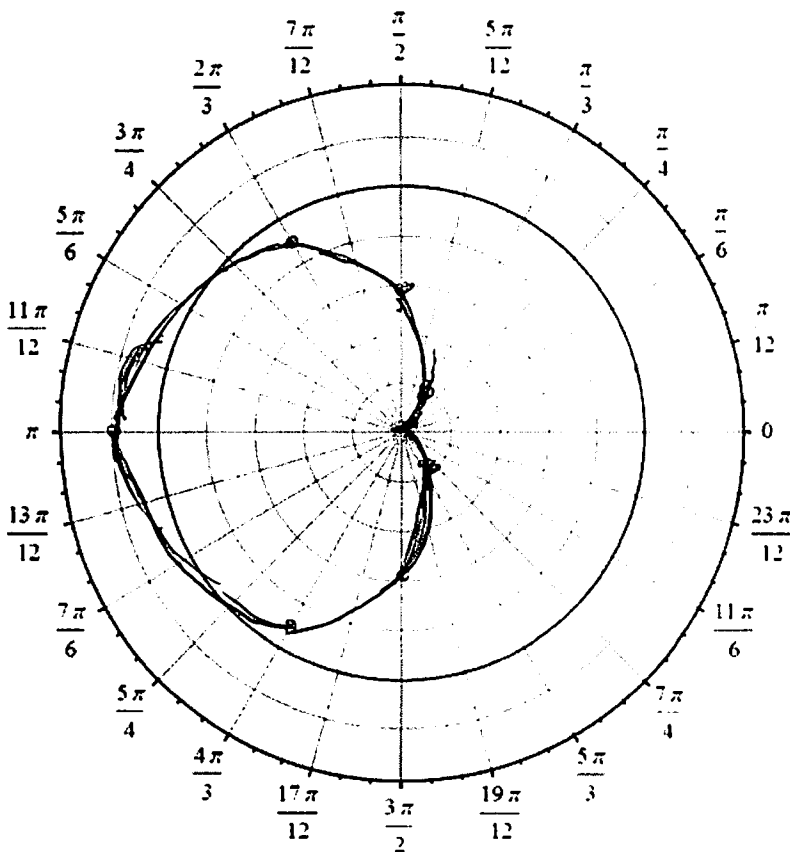
$(x, y)_{\theta = \frac{4\pi}{3}} = (-2.25, -2.25\sqrt{3})$

VTL: $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$(x, y)_{\theta = \frac{\pi}{3}} = (0.75, 0.75\sqrt{3})$

$(x, y)_{\theta = \frac{5\pi}{3}} = (0.75, -0.75\sqrt{3})$

b. Sketch C



$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin\theta + r \cos\theta = 3\sin^2\theta + 3(1-\cos\theta)\cos\theta$$

$$= 3\sin^2\theta + 3\cos\theta - 3\cos^2\theta = 0$$

$$3\cos\theta = 3\cos^2\theta$$

$$\cos\theta = \cos^2\theta$$

$$0 = 2\cos^2\theta - \cos\theta - 1$$

$$0 = (2\cos\theta + 1)(\cos\theta - 1)$$

$$\cos\theta = -\frac{1}{2}, \cos\theta = 1$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos\theta - r \sin\theta$$

$$= 3\sin\theta \cos\theta - 3(1-\cos\theta)\sin\theta$$

$$= 6\sin\theta \cos\theta - 3\sin\theta = 0$$

$$3\sin\theta(2\cos\theta - 1) = 0$$

$$\sin\theta = 0 \quad \cos\theta = \frac{1}{2}$$

$$\theta \neq 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$$

At $\theta = 0$: $\lim_{\theta \rightarrow 0} \frac{3(\cos\theta - \cos 2\theta)}{3\sin\theta(2\cos\theta - 1)}$

$$= \lim_{\theta \rightarrow 0} \frac{(1 + \cos\theta - 2\cos^2\theta)}{\sin\theta(2\cos\theta - 1)}$$

$$= \lim_{\theta \rightarrow 0} \frac{(1 + \cos\theta)(-1 - 2\cos\theta)}{\sin\theta(2\cos\theta - 1)}$$

$$= \lim_{\theta \rightarrow 0} -\frac{(\cos\theta - 1)(2\cos\theta + 1)}{\sin\theta(2\cos\theta - 1)}$$

$$= \lim_{\theta \rightarrow 0} -\frac{\sin\theta}{\cos\theta} \lim_{\theta \rightarrow 0} \frac{2\cos\theta + 1}{2\cos\theta - 1}$$

$$= 0 \cdot 3 = 0$$

So, HTL at $\theta = 0$.

$(x, y)_{\theta=0} = (0, 0)$

HTL at polar:

$(0, 0)_P, (4.5, \frac{2\pi}{3})_P, (4.5, \frac{4\pi}{3})_P$

VTL at polar

$(1.5, \frac{\pi}{3})_P, (1.5, \frac{5\pi}{3})_P, (6, \pi)_P$

Continue from #3: Let $C: r = 3 - 3 \cos \theta$

c. Find the area of the region bounded by C

$$\text{Area} = 2 \cdot \frac{1}{2} \int_0^{\pi} (3 - 3 \cos \theta)^2 d\theta.$$

$$= 9 \int_0^{\pi} (1 - \cos \theta)^2 d\theta$$

$$= 9 \int_0^{\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta$$

$$= 9 \int_0^{\pi} \left(1 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= 9 \int_0^{\pi} \left(\frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta\right) d\theta$$

$$= 9 \cdot \frac{3}{2} \pi$$

$$= \frac{27}{2} \pi$$

$$\underline{\underline{\frac{27}{2} \pi}}$$

d. Find the perimeter of C .

$$\frac{dr}{d\theta} = 3 \sin \theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (3 - 3 \cos \theta)^2 + (3 \sin \theta)^2$$

$$= 9(1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)$$

$$= 9 \cdot 2(1 - \cos \theta)$$

$$= 18 \cdot 2 \sin^2\left(\frac{\theta}{2}\right)$$

$$= 36 \sin^2\left(\frac{\theta}{2}\right)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 6 \sin\left(\frac{\theta}{2}\right)$$

$$P = 2 \int_0^{\pi} 6 \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 12 \left(-2 \cos\left(\frac{\theta}{2}\right)\right) \Big|_0^{\pi}$$

$$= 24$$

$$\underline{\underline{24}}$$

4. (16.6.10) Let curves $\begin{cases} C_1: r = -3 \cos \theta \\ C_2: r = 1 - \cos \theta \end{cases}$

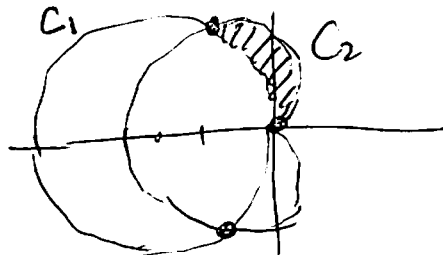
a. Find the angle θ where the curves intersect. Hint: It will be helpful if you sketch the graph

$$-3 \cos \theta = 1 - \cos \theta$$

$$-2 \cos \theta = 1$$

$$\cos \theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$



Also at $\theta_2 = 0$ when $\theta_1 = \frac{\pi}{2}$ (at $(0,0)$)

b. Find the area of the region inside C_2 and outside C_1 .

We will use symmetry

$$\begin{aligned} \text{Area} &= 2 \left[\frac{1}{2} \int_0^{\frac{2\pi}{3}} (1 - \cos \theta)^2 d\theta - \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-3 \cos \theta)^2 d\theta \right] \\ &= \int_0^{\frac{2\pi}{3}} (1 - \cos \theta)^2 d\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 9 \cos^2 \theta d\theta \\ &= \int_0^{\frac{2\pi}{3}} (1 - 2 \cos \theta + \cos^2 \theta) d\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 9 \cos^2 \theta d\theta \\ &= \int_0^{\frac{2\pi}{3}} \left(1 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta - \frac{9}{2} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (1 + \cos 2\theta) d\theta \\ &= \frac{3}{2} \left(\frac{2\pi}{3} \right) - 2 \sin \theta \Big|_0^{\frac{2\pi}{3}} + \frac{1}{4} \sin 2\theta \Big|_0^{\frac{2\pi}{3}} - \frac{9}{2} \left(\frac{\pi}{6} \right) + \frac{9}{4} \sin 2\theta \Big|_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \\ &= \pi - \sqrt{3} - \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{9\sqrt{3}}{8} + 0 \\ &= \frac{\pi}{4} - \frac{\sqrt{3}}{8} \\ &\approx \underline{\underline{0.56889}} \end{aligned}$$

5. (13:4,5,4) Determine whether the sequence converges or diverges, and if it converges, find the limit. Use correct notation, format, explanation, and state any theorems used.

a. $a_n = \frac{\sin n}{n}$

$$|a_n| = \left| \frac{\sin n}{n} \right| \leq \frac{1}{n} \rightarrow 0$$

$\Rightarrow |a_n| \rightarrow 0$ by Squeeze Thm

Since $|a_n| \rightarrow 0$

then $a_n \rightarrow 0$.

c. $a_n = \frac{n^n}{n!} = \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \frac{n}{n-3} \cdots \frac{n}{1}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow$
 $1 \quad >1 \quad >1 \quad >1 \quad \cdots \quad >1 \quad >1$

$$\geq n \rightarrow \infty$$

So, $a_n \rightarrow \infty$ by comparison

b. $a_n = n \tan\left(\frac{1}{n}\right)$

Let $f(x) = x \tan\left(\frac{1}{x}\right)$

note that $f(n) = a_n$.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$t = \frac{1}{x}$

$$= \lim_{t \rightarrow 0^+} \frac{\tan t}{t}$$

$$= 1$$

By CSF, $a_n \rightarrow 1$.

$$\stackrel{LH}{=} \lim_{x \rightarrow \infty} \frac{\sec^2\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{x}\right)$$

$$= \sec^2 0$$

$$= 1$$

6. (7) Find the limit $a_n = \left(1 + \frac{b}{n}\right)^n$

Let $f(x) = \left(1 + \frac{b}{x}\right)^x$, note $f(n) = a_n$.

$$\text{let } y = \left(1 + \frac{b}{x}\right)^x$$

$$\ln y = x \ln \left(1 + \frac{b}{x}\right)$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{b}{x}\right)$$

$$\ln \left(\lim_{x \rightarrow \infty} y \right) = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{b}{x}\right)}{\frac{1}{x}}$$

$$\stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{b}{x}} \cdot \left(-\frac{b}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{b}{1 + \frac{b}{x}}$$

$$= \frac{b}{1+0} = b$$

$$\lim_{x \rightarrow \infty} y = e^b$$

$$\text{So, } a_n \rightarrow e^b$$