Su09 Math 263 Test 2 Name: SOLUTION

<u>Show all necessary work neatly, clearly, systematically, and understandably. Any incorrect statement and/or understatement may be penalized. There are 109 points available.</u>

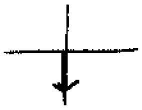
1. (12: 2 each) On a hill, the elevation z meters above a point (x,y) is given by $z = 1500 - 3x^2 - 2y^2$, where (x,y) is in a horizontal xy-plane that lies at the sea level. Positive x-axis points east, positive y-axis points north, both in meters. A climber is at the point (10, -20, 400).

   a. Compute ∇z at (10, -20, 400).

   $\nabla z(x,y) = \langle -6x, -4y \rangle$

   $\nabla z(10,-20) = \underline{\langle -60, 80 \rangle}$

   b. If the climber uses a compass reading to walk due south, by what rate will he ascend or descend?



   $\vec{u} = \langle 0, -1 \rangle$

   $D_{\vec{u}} z(10,-20) = \langle -60, 80 \rangle \cdot \langle 0, -1 \rangle$

   $= 0 - 80 = 80.$

   He will <u>descend</u> by <u>80 m</u>

   c. If the climber uses a compass reading to walk due northeast, by what rate will he ascend or descend?

   $\vec{v} = \langle 1, 1 \rangle$

   $\vec{u}_{\vec{v}} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

   $D_{\vec{u}_{\vec{v}}} z(10,-20) = \langle -60, 80 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = -\frac{60}{\sqrt{2}} + \frac{80}{\sqrt{2}} = \frac{20}{\sqrt{2}} = 10\sqrt{2}$

   He will ascend by <u>$10\sqrt{2}$ m.</u>

   d. In which direction the slope is the largest?

   $\nabla z(10,-20) = \langle -60, 80 \rangle$, $\|\nabla z(10,-20)\| = \sqrt{60^2 + 80^2} = 100$

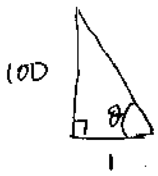
   $\vec{u}_{\nabla z(10,-20)} = \underline{\langle -0.6, 0.8 \rangle}$

   e. At the direction in which the slope is the largest, what is the rate of ascend?

   $\|\nabla z(10,-20)\| = 100$ m.

   At the direction of $\langle -0.6, 0.8 \rangle$, <u>he will ascend by 100 m.</u>

   f. At the direction in which the slope is the largest, at what angle above horizontal does the path in that direction begin?



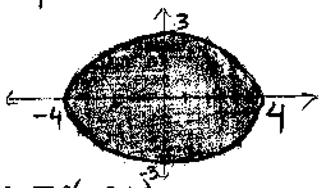
   $\tan\theta = 100$

   $\theta = \underline{\tan^{-1} 100.}$

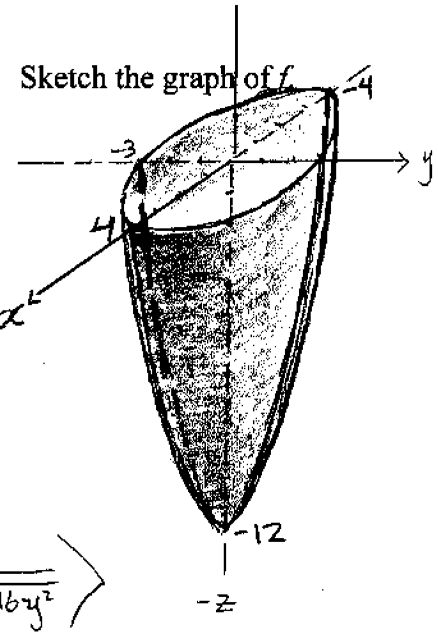
2. (17: 2,3,2,2,2,3,3) Let $f(x,y) = -\sqrt{144-9x^2-16y^2}$.

a. Find and sketch the domain of f.

$$144 - 9x^2 - 16y^2 \geq 0$$
$$144 \geq 9x^2 + 16y^2$$
$$1 \geq \frac{x^2}{16} + \frac{y^2}{9}$$
$$\Rightarrow \text{Dom } f = \left\{ (x,y) \,\middle|\, \frac{x^2}{16} + \frac{y^2}{9} \leq 1 \right\}$$



b. Sketch the graph of f.



c. Find $\nabla f(-3,1)$

$$\nabla f(x,y) = \left\langle \frac{9x}{\sqrt{144-9x^2-16y^2}}, \frac{16y}{\sqrt{144-9x^2-16y^2}} \right\rangle$$
$$\nabla f(-3,1) = \left\langle \frac{-27}{\sqrt{47}}, \frac{16}{\sqrt{47}} \right\rangle$$

d. Find the equation of the plane tangent to the surface of the function at (-3,1).

$$f(-3,1) = -\sqrt{47}$$
$$\Rightarrow \quad z + \sqrt{47} = -\frac{27}{\sqrt{47}}(x+3) + \frac{16}{\sqrt{47}}(y-1)$$
$$\sqrt{47}z + 47 = -27x - 81 + 16y - 16$$
$$\rightarrow \sqrt{47}z = -27x + 16y - 144$$
$$z = \frac{\sqrt{47}}{47}(-27x + 32y - 144)$$

e. Find the equation of the line normal to the surface of the function at (-3,1).

Let $F(x,y,z) = f(x,y) - z$
$$\nabla F(x,y,z) = \langle f_x, f_y, -1 \rangle \big|_{(x,y,z)}$$
$$\nabla F(-3,1,-\sqrt{47}) = \left\langle -\frac{27}{\sqrt{47}}, \frac{16}{\sqrt{47}}, -1 \right\rangle$$
$$\parallel \langle -27, 16, -\sqrt{47} \rangle$$

$$\vec{r}(t) = \langle -3, 1, -\sqrt{47} \rangle + t \langle -27, 16, -\sqrt{47} \rangle$$

f. Linear-approximate $f(-2.94, 1.04)$ from (-3,1).

$$L_z(-2.94, 1.04) = \frac{\sqrt{47}}{47}\left[-27(-2.94) + 16\cdot(1.04) - 144\right]$$
$$= \frac{\sqrt{47}}{47}(-15.34) \cong \underline{-6.9986}$$

g. Find the rate of change of f at point (-3,1) in the direction of $\langle 4,-3 \rangle$.

Let $\vec{v} = \langle 4, -3 \rangle \qquad \|\vec{v}\| = \sqrt{4^2+3^2} = 5$

$$\vec{u}_{\vec{v}} = \langle 0.8, -0.6 \rangle$$
$$D_{\vec{u}_{\vec{v}}} f(-3,1) = \left\langle -\frac{27}{\sqrt{47}}, \frac{16}{\sqrt{47}} \right\rangle \cdot \langle 0.8, -0.6 \rangle$$
$$= \frac{1}{\sqrt{47}}(-27 \cdot 0.8 - 16 \cdot 0.6) = -\frac{31.2}{\sqrt{47}} \cong \underline{-4.55099}$$

3. (14: 2,3,3,3,3) Compute the limits. And if DNE, show two paths with unequal limits.

a. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

① Along $y=0$:

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$

② Along $x=y^2$

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(t^2,t)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{t\to 0} \frac{t^2\cdot t^2}{(t^2)^2+t^4}$

$= \lim_{t\to 0} \frac{t^4}{t^4+t^4} = \lim_{t\to 0} \frac{t^4}{2t^4}$

$= \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$

So, $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ DNE

b. $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^4+y^4+z^2}$

① Along $x=0$; $y=0$

$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^4+y^4+z^2} = \lim_{z\to 0} \frac{0}{z^2} = \lim_{z\to 0} 0 = 0$

② Along $\langle x,y,z\rangle = \langle t, t, t^2\rangle$

$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^4+y^4+z^2} = \lim_{t\to 0} \frac{t\cdot t\cdot t^2}{t^4+t^4+(t^2)^2}$

$= \lim_{t\to 0} \frac{t^4}{t^4+t^4+t^4} = \lim_{t\to 0} \frac{t^4}{3t^4}$

$= \lim_{t\to 0} \frac{1}{3} = \frac{1}{3}$

So, $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^4+y^4+z^2}$ DNE

c. $\lim_{(x,y)\to(0^+,0^+)} \frac{x-y-2\sqrt{x}-2\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$= \lim_{(x,y)\to(0^+,0^+)} \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})-2(\sqrt{x}+\sqrt{y})}{\sqrt{x}+\sqrt{y}}$

$= \lim_{(x,y)\to(0^+,0^+)} \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}-2)}{\sqrt{x}+\sqrt{y}}$

$= \lim_{(x,y)\to(0^+,0^+)} (\sqrt{x}-\sqrt{y}-2)$

$= 0-0-2 = -2$

d. $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

$= \lim_{\substack{r\to 0^+\\ \text{all }\theta}} r^2 \ln r^2$

$= \lim_{\substack{r\to 0^+\\ \text{all }\theta}} \frac{\ln r^2}{r^{-2}} \overset{LH}{=} \lim_{\substack{r\to 0^+\\ \text{all }\theta}} \frac{\frac{2}{r}}{-2r^{-3}}$

$= \lim_{\substack{r\to 0^+\\ \text{all }\theta}} -r^2 = 0$

e. $\lim_{(x,y)\to(0,0)} \frac{x^2\cos y}{x^2+y^4}$

① Along $x=0$:

$\lim_{(x,y)\to(0,0)} \frac{x^2\cos y}{x^2+y^4} = \lim_{y\to 0} \frac{0}{y^4} = \lim_{y\to 0} 0 = 0$

② Along $y=0$

$\lim_{(x,y)\to(0,0)} \frac{x^2\cos y}{x^2+y^4} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$

So, $\lim_{(x,y)\to(0,0)} \frac{x^2\cos y}{x^2+y^4}$ DNE

4. (11: 6,5) Let $S_1: x^2+y^2=5$ and $S_2: x+z=4$ be surfaces in 3-D system.

a. Write the intersection of these surfaces in parametric equation. Hint: First, see the projection on xy-plane.

The projection of S_1 on xy-plane is a circle with radius $\sqrt{5}$ centered at $(0,0)$.

The parametric equation:

$\vec{s}(t) = \langle \sqrt{5}\cos t, \sqrt{5}\sin t \rangle$

In 3-D, it has to satisfy S_2 as well

$z = 4 - x = 4 - \sqrt{5}\cos t$

So, the 3-D parametric equation:

$\vec{r}(t) = \langle \sqrt{5}\cos t, \sqrt{5}\sin t, 4-\sqrt{5}\cos t \rangle$

b. Find the equation of the line tangent to both surfaces at $(1,-2,3)$ $\quad \vec{r}'(t) = \langle -\sqrt{5}\sin t, \sqrt{5}\cos t, \sqrt{5}\sin t \rangle$

(I) At t_0 such that $\vec{r}(t_0) = \langle 1,-2,3 \rangle = \langle \sqrt{5}\cos t, \sqrt{5}\sin t, 4-\sqrt{5}\cos t \rangle$

$\vec{r}'(t_0) = \langle 2, 1, -2 \rangle$

Eq. of line tangent to both surfaces: $\langle 1,-2,3 \rangle + w\langle 2,1,-2 \rangle$

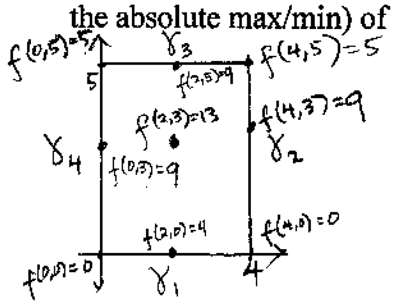
(II) Note that the tangent line is parallel to the cross-product of the normal vectors of the surfaces.

Let $f(x,y,z) = x^2+y^2 \longrightarrow \nabla f = \langle 2x, 2y, 0 \rangle \longrightarrow \nabla f(1,-2,3) = \langle 2,-4,0 \rangle$

$g(x,y,z) = x+z \longrightarrow \nabla g = \langle 1,0,1 \rangle \longrightarrow \nabla g(1,-2,3) = \langle 1,0,1 \rangle$

$\nabla f \times \nabla g \Big|_{(1,-2,3)} = \langle -4,-2,4 \rangle \parallel \langle 2,1,-2 \rangle$ ... then proceed as usual.

5. (15) Find the absolute maximum and absolute minimum values (together with the coordinates that gives the absolute max/min) of $f(x,y) = 4x+6y-x^2-y^2$ over $D = \{(x,y) \mid 0 \le x \le 4, 0 \le y \le 5\}$.



(I) $\nabla f = \langle 4-2x, 6-2y \rangle$

$\nabla f = \vec{0} \Rightarrow x=2, y=3$

$f(2,3) = 8+18-4-9 = 13$

(II) γ_1: $y=0$, $0 \le x \le 5$

$f(x,0) = 4x - x^2$ (← upside down parabola)

min at $(0,0)$ and $(4,0)$

$f(0,0) = f(4,0) = 0$

max at $(2,0)$

$f(2,0) = 4$

γ_2: $x=4$, $0 \le y \le 5$

$f(4,y) = 6y - y^2$

min at $(4,0)$

$f(4,0) = 0$

max at $(4,3)$

$f(4,3) = 9$

γ_3: $y=5$, $0 \le x \le 4$

$f(x,5) = 5 + 4x - x^2$

min at $(0,5)$ and $(4,5)$

$f(0,5) = f(4,5) = 5$

max at $(2,5)$

$f(2,5) = 9$

γ_4: $x=0$, $0 \le y \le 5$

$f(0,y) = 6y - y^2$

(similar to γ_2)

min at $(0,0)$

$f(0,0) = 0$

max at $(0,3)$

$f(0,3) = 9$

So, absolute max

$f(2,3) = 13$

absolute min

$f(0,0) = f(4,0) = 0.$

6. (14: 3,6,5) Let $w = w(x,y,z)$ and $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$. [In other words, w is a function of x,y, and z; and each x,y, and z is a function of u and v].

a. Write the chain-rule for $\dfrac{\partial w}{\partial u}$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

b. Use the result in (a) to find $\dfrac{\partial w}{\partial u}$ if $w = \ln\left(x^2 + y^2 + z^2\right)$ and $(x,y,z) = \left(ue^v \sin u, ue^v \cos u, ue^v\right)$
MAKE SURE YOU SIMPLIFY.

$$\frac{\partial w}{\partial u} = \frac{2x}{x^2+y^2+z^2}\left(e^v \sin u + ue^v \cos u\right) + \frac{2y}{x^2+y^2+z^2}\left(e^v \cos u - ue^v \sin u\right) + \frac{2z\, e^v}{x^2+y^2+z^2}$$

$$= \frac{2e^v}{x^2+y^2+z^2}\left[x \sin u + xu \cos u + y \cos u - yu \sin u + z\right]$$

$$= \frac{2e^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\left[ue^v \sin^2 u + \cancel{u^2e^v \cos u \sin u} + ue^v \cos^2 u - \cancel{u^2e^v \sin u \cos u} + ue^v\right]$$

$$= \frac{2e^v}{u^2e^{2v} + u^2e^{2v}}\left(ue^v \sin^2 u + ue^v \cos^2 u + ue^v\right)$$

$$= \frac{2e^v}{2u^2e^{2v}}\left(ue^v + ue^v\right)$$

$$= \frac{1}{u^2e^v}\, 2ue^v = \frac{2}{u}$$

c. Substitute the x,y, and z in $w = \ln\left(x^2 + y^2 + z^2\right)$ by $(x,y,z) = \left(ue^v \sin u, ue^v \cos u, ue^v\right)$, simplify, and then find $\dfrac{\partial w}{\partial u}$.

$$w = \ln\left(2u^2e^{2v}\right)$$

$$\frac{\partial w}{\partial u} = \frac{1}{2u^2e^{2v}}\frac{\partial\left(2u^2e^{2v}\right)}{\partial u}$$

$$= \frac{1}{2u^2e^{2v}} \cdot 2 \cdot 2u \cdot e^{2v}$$

$$= \frac{2}{u}$$

7. (14: 4,3,7) Consider $f(x,y)=x^4+y^4-4xy$.

a. Find all of its critical points.

$$\nabla f(x,y)=\langle 4x^3-4y,\ 4y^3-4x\rangle$$

$$\nabla f(x,y)=0 \Rightarrow \begin{cases} 4x^3-4y=0 \ \dots ① \\ 4y^3-4x=0 \ \dots ② \end{cases}$$

① $4x^3-4y=0$, $4(x^3-y)=0$, $x^3=y$

② $4y^3-4x=0$, $4(y^3-x)=0$, $y^3=x$

①⇒②: $(x^3)^3=x$

$x^9=x$

$x^9-x=0$

$x(x^8-1)=0$

$x(x^4+1)(x^4-1)=0$

$x(x^4+1)(x^2+1)(x^2-1)=0$

$x(x^4+1)(x^2+1)(x+1)(x-1)=0$

$x_1=0$, $y_1=0$; $x_2=-1$, $y_2=-1$; $x_3=1$, $y_3=1$ ⇒ Critical points: $(0,0)$

b. Find $f_{xx}(x,y), f_{yy}(x,y), f_{xy}(x,y)$.

$f_{xx}(x,y)=12x^2$

$f_{xy}(x,y)=-4$ $\qquad f_{yy}(x,y)=12y^2$

c. Find the Hessian D of each critical point and then specify each critical point as local max, local min, or saddle point.

$$D(x,y)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy}\end{vmatrix}=12x^2\cdot 12y^2-(-4)^2=144x^2y^2-16$$

$D(0,0)=144\cdot 0-16=-16<0 \longrightarrow (0,0)$ is a saddle point

$D(1,1)=144\cdot1\cdot1-16=128>0$ since $f_{xx}(1,1)=12>0$ $\Big\}\Rightarrow (1,1)$ is a local min

$D(-1,-1)=144\cdot1\cdot1-16=128>0$ since $f_{xx}(-1,-1)=12>0$ $\Big\}\Rightarrow (-1,-1)$ is a local min

8. (12) Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex on the first octant of $3x+2y+z=6$. First define the function you are about to maximize in terms of x and y only.

$\rightarrow z=6-3x-2y$

Volume $=xyz$

$V(x,y,z)=xyz$

$V(x,y)=xy(6-3x-2y)$

$=6xy-3x^2y-2xy^2$

$V_x(x,y)=6y-6xy-2y^2$

$V_y(x,y)=6x-3x^2-4xy$

$$\nabla V=\vec{0} \Rightarrow \begin{cases} 6y-6xy-2y^2=0 \\ 6x-3x^2-4xy=0\end{cases}$$

$$\Rightarrow \begin{cases} 2y(3-3x-y)=0 \dots ① \\ x(6-3x-4y)=0 \dots ②\end{cases}$$

①: $2y(3-3x-y)=0$

$\rightarrow y=0$ or $3x+y=3$

②: $x(6-3x-4y)=0$

$\rightarrow x=0$ or $3x+4y=6$

Now, for $x=0$ or $y=0$: Volume $=0$

for $\begin{cases} 3x+y=3 \\ 3x+4y=6 \end{cases}$ $-$

$-3y=-3$

$y=1 \rightarrow x=\frac{2}{3}$

$V(\frac{2}{3},1)=\frac{2}{3}\cdot1\,(6-3\cdot\frac{2}{3}-2\cdot1)$

$=\frac{2}{3}\cdot1\cdot(6-2-2)$

$=\frac{2}{3}\cdot1\cdot2=\frac{4}{3}$

So, the maximum volume is $\frac{4}{3}$