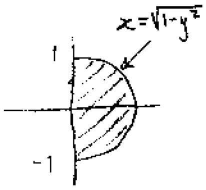
Su09 Math 263 Test 3 Name: SOLUTION

Show all necessary work neatly, clearly, systematically, and understandably. Any incorrect statement and/or understatement may be penalized. There are 106 points available.

1. (6) Evaluate: $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} \left(2+x^2+y^2\right)^{5/2} dydx$.



$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} (2+r^2)^{\frac{5}{2}}\, r\, dr d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cdot \int_{0}^{1} (2+r^2)^{\frac{5}{2}} r\, dr$

$= \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \int_0^1 (2+r^2)^{\frac{5}{2}} \cancel{r} \frac{d(2+r^2)}{2\cancel{r}}$

$= \left(\frac{\pi}{2}+\frac{\pi}{2}\right) \cdot \frac{1}{2}\int_0^1 (2+r^2)^{\frac{5}{2}} \cdot d(2+r^2)$

$= \pi \cdot \frac{1}{2} \cdot \frac{2}{7} \left(2+r^2\right)^{\frac{7}{2}}\Big|_0^1$

$= \frac{\pi}{7}\left(3^{\frac{7}{2}} - 2^{\frac{7}{2}}\right)$

2. (6) Evaluate: $\int_{0}^{1}\int_{2y}^{2} e^{x^2}\, dxdy$

$= \int_0^2 \int_0^{\frac{1}{2}x} e^{x^2}\, dy\, dx$

$= \int_0^2 e^{x^2} y \Big|_0^{\frac{x}{2}} dx$

$= \int_0^2 e^{x^2} \cdot \frac{x}{2}\, dx$

$= \frac{1}{2}\int_0^2 e^{x^2} \cdot x \frac{dx^2}{2x}$

$= \frac{1}{4}\int_0^2 e^{x^2} dx^2$

$= \frac{1}{4} e^{x^2}\Big|_0^2$

$= \frac{1}{4}\left(e^4 - e^0\right) = \frac{1}{4}\left(e^4 - 1\right)$

3. (10) Use Lagrange Multiplier method to find the greatest and smallest values that the function $f(x,y) = 4x^3 + y^2$ takes on $2x^2+y^2=1$. Note: You need to find ALL critical points of this method.

Let $g(x,y) = 2x^2+y^2 = 1$

$\nabla f = \langle 12x^2, 2y\rangle$

$\nabla g = \langle 4x, 2y\rangle$

$\Rightarrow \nabla f = \lambda \nabla g$

$\Rightarrow \begin{cases} 12x^2 = 4\lambda x & \cdots ① \\ 2y = 2\lambda y & \cdots ② \\ 2x^2+y^2 = 1 & \cdots ③ \end{cases}$

①: $12x^2 = 4\lambda x$
$\to x = 0$ or $\lambda = 3x$

②: $2y = 2\lambda y$
$\to y = 0$ or $\lambda = 1$

If $x=0,\ y=0$: $2(0)^2+0^2 = 1 \to$ contradict ③

If $x=0$: $2(0)^2+y^2 = 1 \to y=\pm 1$
$\to (0,\pm 1)$

If $y=0$: $2x^2+0^2=1$
$\to x = \pm\frac{\sqrt{2}}{2} \to \left(\pm\frac{\sqrt{2}}{2}, 0\right)$

If $\lambda = 1 \to x = \frac{1}{3} \xrightarrow{③} 2\left(\frac{1}{3}\right)^2+y^2=1$

$\frac{2}{9}+y^2=1$

$y^2 = \frac{7}{9}$

$y = \pm\frac{\sqrt{7}}{3}$

$\to \left(\frac{1}{3}, \pm\frac{\sqrt{7}}{3}\right)$

So, the critical points are $(0,\pm 1)$
$\left(\pm\frac{\sqrt{2}}{2}, 0\right)$
$\left(\frac{1}{3}, \pm\frac{\sqrt{7}}{3}\right)$

$f(0,\pm 1) = 1$

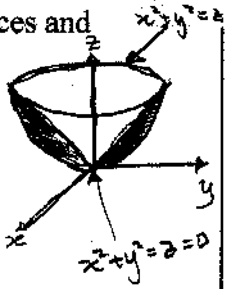
$f\left(\frac{1}{3}, \pm\frac{\sqrt{7}}{3}\right) = \frac{4}{27}+\frac{7}{9} = \frac{25}{27}$

$f\left(\frac{\sqrt{2}}{2}, 0\right) = \sqrt{2} \leftarrow$ max

$f\left(-\frac{\sqrt{2}}{2}, 0\right) = -\sqrt{2} \leftarrow$ min

4. (16: 3,5,8) Consider the solid bounded by $z = x^2 + y^2$ and $z^2 = x^2 + y^2$.

a. Find the intersection the surfaces and sketch the solid.



$$\underset{①}{z} = x^2+y^2 = \underset{②}{z^2}$$

$$z = z^2$$

$$z=0 \qquad z=1$$

$$z=x^2+y^2=0 \qquad x^2+y^2=z=1$$

b. Find the volume of the solid.

below: $z=x^2+y^2=r^2$

above: $z^2=x^2+y^2=r^2$, $z=r$

$$\int_0^{2\pi}\int_0^1 (r-r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^2 - r^3\, dr$$

$$= \theta\Big|_0^{2\pi} \cdot \left(\tfrac{1}{3}r^3 - \tfrac{1}{4}r^4\right)\Big|_0^1$$

$$= 2\pi \cdot \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 2\pi \cdot \tfrac{1}{12}$$

$$= \frac{\pi}{6}$$

c. Find the surface area of the solid.

$$SA = SA_{above} + SA_{below}$$

SA_{above}: $z^2 = x^2+y^2$

$2zz_x = 2x \qquad 2zz_y = 2y$

$z_x = \frac{x}{z} \qquad z_y = \frac{y}{z}$

Integrand: $\sqrt{1+z_x^2+z_y^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}$

$= \sqrt{1+\frac{x^2+y^2}{z^2}}$

$= \sqrt{1+\frac{x^2+y^2}{x^2+y^2}}$

$= \sqrt{2}$

$$SA_{above} = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\,dr\,d\theta$$

$$= \sqrt{2}\underbrace{\int_0^{2\pi}\int_0^1 r\,dr\,d\theta}_{\text{area of circle}}$$

$$= \sqrt{2}\,\pi$$

SA_{below}: $z = x^2+y^2$

$z_x = 2x \qquad z_y = 2y$

in polar

Integrand: $\sqrt{1+z_x^2+z_y^2} = \sqrt{1+4x^2+4y^2} = \sqrt{1+4r^2}$

$$SA_{below} = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^1 \sqrt{1+4r^2}\; r\,\frac{d(1+4r^2)}{8r}$$

$$= \frac{\pi}{4}\int_0^1 (1+4r^2)^{\frac{1}{2}}\, d(1+4r^2)$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}\,(1+4r^2)^{\frac{3}{2}}\Big|_0^1 = \frac{\pi}{6}(5^{\frac{3}{2}}-1)$$

So, $SA = \sqrt{2}\pi + \frac{1}{6}\pi(5\sqrt{5}-1)$

5. (10) Use Lagrange Multiplier method to find the dimension of the box with greatest volume, provided the surface area is 10 m^2. First define the function you need to maximize and also define the constraint.

max $V(x,y,z) = xyz \qquad x,y,z \geq 0$

Constraint: $2xy + 2xz + 2yz = 10$

$g(x,y,z) = xy+xz+yz = 5$

$\nabla V = \langle yz, xz, xy\rangle$

$\nabla g = \langle y+z, x+z, x+y\rangle$

$\nabla V = \lambda \nabla g$

$$\begin{cases} yz = \lambda(y+z) & \cdots ① \\ xz = \lambda(x+z) & \cdots ② \\ xy = \lambda(x+y) & \cdots ③ \\ xy+xz+yz = 5 & \cdots ④ \end{cases}$$

NOTE: If $x=0$

then ②: $0 = \lambda z$

$\lambda = 0$ or $z=0$

if $\lambda = 0$ ①: $yz=0$

$y=0$ or $z=0$

Either $x=y=0$ or $x=z=0$

$xy+xz+yz=0$ (contradict ④)

so, $x,y,z > 0$

Since $x,y,z > 0$, then

$$\lambda \overset{①}{=} \frac{yz}{y+z} \overset{②}{=} \frac{xz}{x+z} \overset{③}{=} \frac{xy}{x+y}$$

$yz(x+z) = xz(y+z)$

$xyz + yz^2 = xyz + xz^2$

$yz^2 = xz^2$

$x = y$

$xz(x+y) = xy(x+z)$

$x^2z + xyz = x^2y + xyz$

$x^2z = x^2y$

$y = z$

$\longrightarrow x = y = z$

④: $x^2+x^2+x^2 = 5$

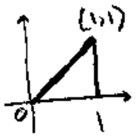
$3x^2 = 5$

$x = \sqrt{\frac{5}{3}} = \frac{\sqrt{15}}{3}$

So, $(x,y,z) = \left(\frac{\sqrt{15}}{3}, \frac{\sqrt{15}}{3}, \frac{\sqrt{15}}{3}\right)$

Volume $= \frac{15\sqrt{15}}{27} = \frac{5\sqrt{15}}{9}$

6. (10: 4,3,3) Consider a triangle with vertices (0,0), (1,0), and (1,1) with density proportional to the square of distance from origin. Find mass, M_x and I_x.



$\rho(x,y) = x^2+y^2$

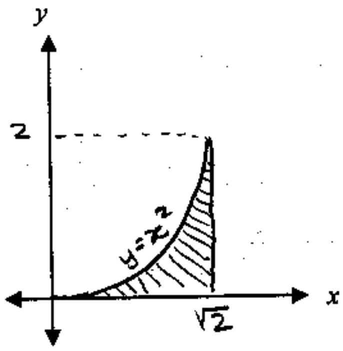
$$m = \int_0^1\int_0^x x^2+y^2\,dy\,dx = \int_0^1 \left(x^2y+\frac{y^3}{3}\right)\Big|_0^x dx = \int_0^1 x^3+\frac{x^3}{3}\,dx = \frac{4}{3}\int_0^1 x^3\,dx = \frac{1}{3}x^4\Big|_0^1 = \frac{1}{3}$$

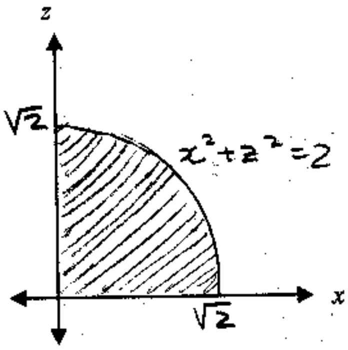
$$M_x = \int_0^1\int_0^x y(x^2+y^2)\,dy\,dx = \int_0^1\int_0^x x^2y+y^3\,dy\,dx = \int_0^1\left(\frac{1}{2}x^2y^2+\frac{1}{4}y^4\right)\Big|_0^x dx = \int_0^1 \frac{1}{2}x^4+\frac{1}{4}x^4\,dx = \frac{3}{4}\int_0^1 x^4\,dx = \frac{3}{20}x^5\Big|_0^1 = \frac{3}{20}$$

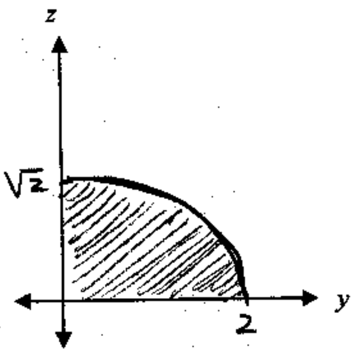
$$I_x = \int_0^1\int_0^x y^2(x^2+y^2)\,dy\,dx = \int_0^1\int_0^x x^2y^2+y^4\,dy\,dx = \int_0^1\left(\frac{1}{3}x^2y^3+\frac{1}{5}y^5\right)\Big|_0^x dx = \int_0^1 \frac{1}{3}x^5+\frac{1}{5}x^5\,dx = \frac{8}{15}\int_0^1 x^5\,dx = \frac{4}{45}x^6\Big|_0^1 = \frac{4}{45}$$

7. (18:6,12) Consider a solid in the first octant bounded by $y = x^2$, and $z^2 + x^2 = 2$.

a. (1,1,4) Sketch and write the equation of the projection of the solid onto *xy*-plane, *xz*-plane and *xz*-plane.







$y = x^2$

$z^2+x^2 = 2 \xrightarrow{z=0} x^2=2$

FIRST OCTANT $\longrightarrow x=\sqrt{2}$

$\left.\begin{array}{l} y=x^2 \\ z^2+x^2=2 \end{array}\right\} \Longrightarrow z^2+y=2$

$y = 2-z^2$

b. Write six different triple integrals that gives the volume of the solid. You don't need to compute.

$$\int_0^{\sqrt{2}}\int_0^{x^2}\int_0^{\sqrt{2-x^2}} dz\,dy\,dx \qquad \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_0^{x^2} dy\,dz\,dx \qquad \int_0^{2}\int_0^{\sqrt{2-y}}\int_{\sqrt{y}}^{\sqrt{2-z^2}} dx\,dz\,dy$$

$$\int_0^{2}\int_{\sqrt{y}}^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} dz\,dx\,dy \qquad \int_0^{\sqrt{2}}\int_0^{\sqrt{2-z^2}}\int_0^{x^2} dy\,dx\,dz \qquad \int_0^{\sqrt{2}}\int_0^{2-z^2}\int_{\sqrt{y}}^{\sqrt{2-z^2}} dx\,dy\,dz$$

8. (14: ~~3,5,8~~ 6,8) Consider a solid in the first octant, $\rho(x,y,z) = xyz$, bounded by $y = x^2$ and $z^2 + x^2 = 2$

a. Find the volume. NOTE: Same solid to #7

$$V = \int_0^{\sqrt{2}}\int_0^{x^2}\int_0^{\sqrt{2-x^2}} dz\,dy\,dx$$

$$= \int_0^{\sqrt{2}}\int_0^{x^2} \sqrt{2-x^2}\,dy\,dx$$

$$= \int_0^{\sqrt{2}} \sqrt{2-x^2}\cdot x^2\,dx$$

$x = \sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\,d\theta$

$\sqrt{2-x^2} = \sqrt{2-2\sin^2\theta} = \sqrt{2}\cos\theta$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\cos\theta\cdot(\sqrt{2}\sin\theta)^2\sqrt{2}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right]$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

Alternatively:

$$\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_0^{x^2} dy\,dz\,dx$$

$$= \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} x^2\,dz\,dx$$

$\left[\text{Convert } xz \text{ to polar } \begin{array}{l} x = r\cos\theta \\ z = r\sin\theta \end{array}\right]$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} r^2\cos^2\theta\; r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta\int_0^{\sqrt{2}} r^3\,dr$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta\cdot\left(\frac{1}{4}r^4\right)\Big|_0^{\sqrt{2}}$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}\cdot\frac{1}{4}(4-0)$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]\cdot 1$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

b. Find the mass of the solid.

$$m = \int_0^{\sqrt{2}}\int_0^{x^2}\int_0^{\sqrt{2-x^2}} xyz\,dz\,dy\,dx$$

$$= \int_0^{\sqrt{2}}\int_0^{x^2} \frac{xyz^2}{2}\Big|_0^{\sqrt{2-x^2}}\,dy\,dx$$

$$= \frac{1}{2}\int_0^{\sqrt{2}}\int_0^{x^2} xy(2-x^2)\,dy\,dx$$

$$= \frac{1}{4}\int_0^{\sqrt{2}} xy^2(2-x^2)\Big|_0^{x^2}\,dx$$

$$= \frac{1}{4}\int_0^{\sqrt{2}} x\cdot x^4(2-x^2)\,dx$$

$$= \frac{1}{4}\int_0^{\sqrt{2}} 2x^5 - x^7\,dx$$

$$= \frac{1}{4}\left(\frac{1}{3}x^6 - \frac{1}{8}x^8\right)\Big|_0^{\sqrt{2}}$$

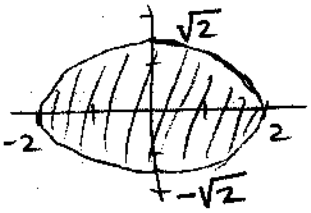
$$= \frac{1}{4}\left(\frac{1}{3}\cdot 8 - \frac{1}{8}\cdot 16\right)$$

$$= \frac{1}{4}\left(\frac{8}{3}-2\right) = \frac{1}{4}\cdot\frac{2}{3} = \frac{1}{6}$$

9. (16: 4,4,8) Let a solid be bounded by $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

a. Find the projection of intersection of the surfaces onto xy-plane. You need this for the domain of integration.

$$z = x^2 + 3y^2$$
$$-\quad z = 8 - x^2 - y^2$$
$$0 = 2x^2 + 4y^2 - 8$$
$$x^2 + 2y^2 = 4$$
$$\frac{x^2}{4} + \frac{y^2}{2} = 1$$



b. Set up the double integral for the surface area of the solid. Don't need to compute.

Above: $z = 8 - x^2 - y^2$, $z_x = -2x$, $z_y = -2y$ $\longrightarrow$ Integrand $= \sqrt{1+4x^2+4y^2}$

Below: $z = x^2 + 3y^2$, $z_x = 2x$, $z_y = 6y$ $\longrightarrow$ Integrand $= \sqrt{1+4x^2+36y^2}$

$$SA = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \sqrt{1+4x^2+4y^2} + \sqrt{1+4x^2+36y^2}\; dy\,dx$$

$$\cos^4\theta = (\cos^2\theta)^2 = \left(\frac{1+\cos 2\theta}{2}\right)^2$$

c. Find the volume of the solid. Note: Can't be done by polar.

$$V = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-x^2-y^2) - (x^2+3y^2)\, dy\,dx$$

by symmetry

$$= 4\int_0^2\int_0^{\sqrt{\frac{4-x^2}{2}}} 8 - 2x^2 - 4y^2\, dy\,dx$$

$$= 4\cdot 2\int_0^2\int_0^{\frac{\sqrt{4-x^2}}{\sqrt{2}}} 4 - x^2 - 2y^2\, dy\,dx$$

$$= 8\int_0^2 \left[(4-x^2)y - \frac{2}{3}y^3\right]_0^{\frac{\sqrt{4-x^2}}{\sqrt{2}}} dx$$

$$= 8\int_0^2 (4-x^2)\frac{\sqrt{4-x^2}}{\sqrt{2}} - \frac{2}{3}\left(\frac{\sqrt{4-x^2}}{\sqrt{2}}\right)^3 dx$$

$$= 8\int_0^2 \frac{(4-x^2)^{\frac{3}{2}}}{\sqrt{2}} - \frac{(4-x^2)^{\frac{3}{2}}}{3\sqrt{2}}\, dx$$

$$= 8\left(\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}}\right)\int_0^2 (4-x^2)^{\frac{3}{2}}\, dx$$

$$= 8\cdot\frac{\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} (4\cos^2\theta)^{\frac{3}{2}}\cdot 2\cos\theta\, d\theta \qquad \boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \\ 4-x^2 &= 4\cos^2\theta \end{aligned}}$$

$$= \frac{8\sqrt{2}}{3}\cdot 16\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= \frac{128\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{32\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} 1 + 2\cos 2\theta + \cos^2 2\theta\, d\theta$$

$$= \frac{32\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{32\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\, d\theta$$

$$= \frac{32\sqrt{2}}{3}\left[\frac{3}{2}\theta + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{32\sqrt{2}}{3}\left[\left(\frac{3\pi}{4} + 0 + 0\right) - (0+0+0)\right]$$

$$= \frac{32\sqrt{2}}{3}\cdot\frac{3\pi}{4}$$

$$= \underline{\underline{8\sqrt{2}\,\pi}}$$