Su09 Math 263 Test 4 Name: SOLUTION

<u>Show all necessary work neatly, clearly, systematically, and understandably. Any incorrect statement and/or understatement may be penalized. There are 105 points available.</u>

1. (10:7,3) Let $\vec{F} = \langle y\cos xy, x\cos xy, z\sin yz \rangle$. Find:

   a. *Curl* $\vec{F}$. Note: Show the determinant!

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos xy & x\cos xy & z\sin yz \end{vmatrix} = \langle z^2\cos yz - 0, -(0-0), (\cos xy - xy\sin xy) - (\cos xy - xy\sin xy) \rangle$$

$$= \langle z^2 \cos yz, 0, 0 \rangle$$

   b. *Div* $\vec{F}$ $= \dfrac{\partial(y\cos xy)}{\partial x} + \dfrac{\partial(x\cos xy)}{\partial y} + \dfrac{\partial(z\sin yz)}{\partial z}$

$$= -y^2\sin xy - x^2 \sin xy + \sin yz + yz\cos yz$$

2. (14:5,9) A thin wire in the shape of a quarter-circle of radius 2 of the first quadrant with the density function is $\rho(x,y) = x + y$.

   a. Find the mass.

$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle,\ 0 \le t \le \frac{\pi}{2}$

$\vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle$

$\|\vec{r}'(t)\| = 2$

$$= \int_C (x+y)\,ds = \int_0^{\frac{\pi}{2}} (2\cos t + 2\sin t)\, 2\,dt = 4\int_0^{\frac{\pi}{2}} \cos t + \sin t\, dt$$

$$= 4(\sin t - \cos t)\Big|_0^{\frac{\pi}{2}} = 4\big((1-0)-(0-1)\big) = 4\cdot 2 = 8$$

   b. Find the center of mass. Note: the situation is symmetric in x and y.

$$M_y = \int_C x\rho\,ds = \int_0^{\frac{\pi}{2}} (2\cos t)(2\cos t + 2\sin t)\,2\,dt = 4\int_0^{\frac{\pi}{2}} 2\cos^2 t + 2\cos t\sin t\,dt$$

$$= 4\int_0^{\frac{\pi}{2}} \underline{1+\cos 2t} + \underline{\sin 2t}\,dt = 4\left(t + \frac{\sin 2t}{2} - \frac{\cos 2t}{2}\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 4\left[\left(\frac{\pi}{2} + 0 + \frac{1}{2}\right) - \left(0 + 0 - \frac{1}{2}\right)\right] = 4\left[\frac{\pi}{2} + 1\right] = 2(\pi + 2)$$

$$\bar{x} = \frac{M_y}{m} = \frac{2(\pi+2)}{8} = \frac{\pi+2}{4}$$

By symmetry $\bar{y} = \dfrac{\pi+2}{4}$

So, the center of mass $(\bar{x}, \bar{y}) = \left(\dfrac{\pi+2}{4}, \dfrac{\pi+2}{4}\right)$

3. (8) Compute: $\int_C \vec{F}\cdot d\vec{r}$. where $\vec{F}=\langle z,y,-x\rangle$ and $\vec{r}(t)=\langle t,\sin t,\cos t\rangle, 0\le t\le \pi$

$$d\vec{r} = \langle 1, \cos t, -\sin t\rangle dt$$

$$= \int_C \langle z,y,-x\rangle \cdot \langle 1, \cos t, -\sin t\rangle dt$$

$$= \int_0^\pi \langle \cos t, \sin t, -t\rangle \cdot \langle 1, \cos t, -\sin t\rangle dt$$

$$= \int_0^\pi \cos t + \sin t \cos t + t \sin t \, dt$$

$$= \left(\cancel{\sin t} + \cancel{\frac{\sin^2 t}{2}} - t\cos t + \cancel{\sin t}\right)\Big|_0^\pi$$

$$= -\pi \cos \pi = \underline{\underline{\pi}}$$

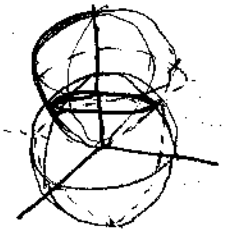
ASIDE

$$\int t \sin t\, dt = -t\cos t + \sin t + C \qquad \begin{array}{c|l} t & \sin t \\ 1 & -\cos t \\ 0 & -\sin t \end{array}$$

4. (10) Find the mass of solid $R=\{(x,y,z) \mid 4\le x^2+y^2+z^2\le 4z\}$ if the density is inversely proportional to the square distance from the origin.

$$4 \le \rho^2 \le 4\rho\cos\varphi \qquad \text{density} = \frac{k}{x^2+y^2+z^2} = \frac{k}{\rho^2}$$

$$\rightarrow \begin{cases} 4 = \rho^2 \rightarrow \rho = 2 \\ \rho^2 = 4\rho\cos\varphi \rightarrow \rho = 4\cos\varphi \end{cases} \rightarrow 2 = 4\cos\varphi, \quad 0.5 = \cos\varphi, \quad \varphi = \frac{\pi}{3}$$



$$= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_2^{4\cos\varphi} \frac{k}{\cancel{\rho^2}} \cancel{\rho^2} \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 2k\pi \int_0^{\frac{\pi}{3}} 4\sin\varphi\cos\varphi - 2\sin\varphi \, d\varphi$$

$$= 2k\pi \int_0^{\frac{\pi}{3}} 2\sin 2\varphi - 2\sin\varphi \, d\varphi$$

$$= 4k\pi \left(-\frac{\cos 2\varphi}{2} + \cos\varphi\right)\Big|_0^{\frac{\pi}{3}}$$

$$= 4k\pi\left[\left(\frac{1}{4}+\frac{1}{2}\right) - \left(-\frac{1}{2}+1\right)\right]$$

$$= 4k\pi\left(\frac{1}{4}\right)$$

$$= \underline{\underline{k\pi}}$$

5. (10) Compute: $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=\left\langle e^x \ln y - \frac{e^y}{x}, \frac{e^x}{y} - e^y \ln x, 2z\right\rangle$ and $C: \vec{r}(t)=\langle t,t,t^2\rangle, 1\le t\le 3$.

Hint: the vector field is conservative.

$\vec{F}$ is conservative $\rightarrow$ there is a potential f such that $\vec{F} = \nabla f$

$$\Rightarrow \begin{cases} f_x = e^x \ln y - \frac{e^y}{x} & \rightarrow f = e^x \ln y - e^y \ln x + g_1(y,z) \\ f_y = \frac{e^x}{y} - e^y \ln x & \rightarrow f = e^x \ln y - e^y \ln x + g_2(x,z) \\ f_z = 2z & \rightarrow f = z^2 + g_3(x,y) \end{cases}$$

$$\Rightarrow f(x,y,z) = e^x \ln y - e^y \ln x + z^2$$

$$\int_C \vec{F}\cdot d\vec{r} \overset{\text{FTLI}}{=} f(\vec{r}(3)) - f(\vec{r}(1)) = f(3,3,9) - f(1,1,1)$$

$$= \left(\cancel{e^3 \ln 3} - \cancel{e^3 \ln 3} + 9^2\right) - \left(\cancel{e^1 \ln 1} - \cancel{e^1 \ln 1} + 1^2\right)$$

$$= 81 - 1 = \underline{\underline{80}}$$

ALTERNATE: $\vec{F}(\vec{r}(t)) = \left\langle e^t \ln t - \frac{e^t}{t}, \frac{e^t}{t} - e^t \ln t, 2t^2\right\rangle$

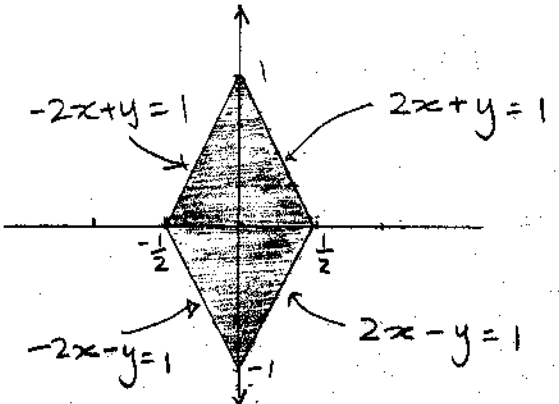
$$d\vec{r}(t) = \langle 1, 1, 2t\rangle dt$$

$$\vec{F}(\vec{r}(t))\cdot d\vec{r} = e^t \ln t - \frac{e^t}{t} + \frac{e^t}{t} - e^t \ln t + 4t^3 = 4t^3$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^3 4t^3\, dt = t^4\Big|_1^3 = 3^4 - 1^4 = \underline{\underline{80}}$$

6. (18:4,3,3,2,6) Let $R = \{(x,y) \mid |2x| + |y| \le 1\}$.

a. Sketch the domain on *xy*-plane.



b. Define transformation *T*.

$$T^{-1}\begin{cases} u = 2x+y \\ v = 2x-y \end{cases}$$

$$u+v = 4x$$
$$u-v = 2y$$

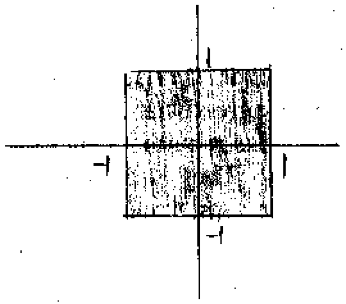
$$T\begin{cases} x = \frac{1}{4}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

c. Compute the Jacobian of T, J_T.

$$J_{T^{-1}} = \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = -2-2 = -4$$

$$J_T = -\frac{1}{4}$$

d. Sketch the domain on *uv*-plane.



e. Now, compute using change of variable: $\iint_R e^{4x}\,dA$.

$$u+v = 4x$$

$$|J_T| = \left|-\frac{1}{4}\right| = \frac{1}{4}$$

$$\iint_R e^{4x}\,dA = \int_{-1}^{1}\int_{-1}^{1} e^{u+v}\,\frac{1}{4}\,du\,dv$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} e^u e^v\,du\,dv$$

$$= \frac{1}{4}\int_{-1}^{1} e^u\,du \cdot \int_{-1}^{1} e^v\,dv$$

$$= \frac{1}{4}\left(\int_{-1}^{1} e^u\,du\right)^2$$

$$= \frac{1}{4}\left(e^u\Big|_{-1}^{1}\right)^2$$

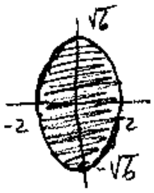
$$= \frac{1}{4}\left(e^1 - e^{-1}\right)^2$$

$$= \frac{1}{4}\left(e^2 - 2 + e^{-2}\right)$$

7. (15) Compute: $\oint_C (3e^x - 2y^3)dx + (4\cos y + 3x^3)dy$, where C: Counter-clockwise path on the boundary of $D = \{(x,y) \mid 3x^2 + 2y^2 = 12\}$.

$$\oint_C \underbrace{(3e^x - 2y^3)}_{P}dx + \underbrace{(4\cos y + 3x^3)}_{Q}dy$$

$$\overset{G}{=} \iint_D 9x^2 + 6y^2 \, dA$$



Now, do change of variable

$$T \begin{cases} x = 2r\cos t & 0 \le r \le 1 \\ y = \sqrt{6}\, r \sin t & 0 \le t \le 2\pi \end{cases}$$

$$J_T = \begin{vmatrix} 2\cos t & -2r\sin t \\ \sqrt{6}\sin t & \sqrt{6} r\cos t \end{vmatrix} = 2\sqrt{6}\, r \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix}$$

$$= 2\sqrt{6}\, r(\cos^2 t + \sin^2 t) = 2\sqrt{6}\, r$$

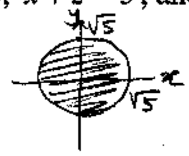
$$9x^2 + 6y^2 = 9(2r\cos t)^2 + 6(\sqrt{6}\, r\sin t)^2$$
$$= 36r^2\cos^2 t + 36r^2 \sin^2 t = 36r^2$$

So, $\iint_D 9x^2 + 6y^2 dA$

$$= \int_0^{2\pi}\int_0^1 36r^2 \cdot 2\sqrt{6}\, r \, dr\, d\theta$$
$$= 72\sqrt{6} \cdot 2\pi \int_0^1 r^3 dr$$
$$= 72\sqrt{6} \cdot 2\pi \cdot \tfrac{1}{4} r^4 \Big|_0^1$$
$$= 36\sqrt{6}\pi (1^4 - 0^4)$$
$$= \underline{\underline{36\sqrt{6}\pi}}$$

8. (20:6,14) Consider a solid bounded by planes $z = 0$, $x + z = 5$, and by cylinder $\underbrace{x^2 + y^2}_{r^2} = 5$.

a. Find the volume of the solid.



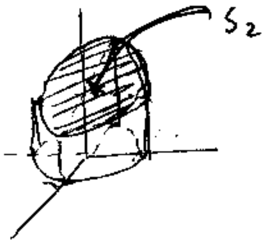
$0 \le r \le \sqrt{5}$
$0 \le \theta \le 2\pi$
$0 \le z \le 5 - x$

$$\int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{5 - r\cos\theta} r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{5}} 5r - r^2\cos\theta \, dr\, d\theta = \int_0^{2\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}r^3\cos\theta\right)\Big|_0^{\sqrt{5}} d\theta = \int_0^{2\pi} \frac{25}{2} - \frac{5\sqrt{5}}{3}\cos\theta \, d\theta$$

$$= \left(\frac{25}{2}\theta - \frac{5\sqrt{5}}{3}\sin\theta\right)\Big|_0^{2\pi} = \frac{25}{2}\cdot 2\pi = \underline{\underline{25\pi}}$$

b. Find the surface area of the solid.



S_1: $x^2 + y^2 \le 5$, $z = 0$

Area(S_1) = Area (Disk, $r = \sqrt{5}$)
$= 5\pi$

S_2: $z = 5 - x$, $x^2 + y^2 \le 5$

$z_x = -1$, $z_y = 0$

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 1 + 0} = \sqrt{2}$$

$$\text{Area}(S_2) = \iint_{S_1} \sqrt{2}\, dA$$
$$= \sqrt{2}\iint_{S_1} dA$$
$$= 5\sqrt{2}\pi$$

S_3: $x^2 + y^2 = 5$, $0 \le z \le 5 - x$

$\vec{r}(t) = \langle \sqrt{5}\cos t, \sqrt{5}\sin t \rangle$

$\vec{r}\,'(t) = \langle -\sqrt{5}\sin t, \sqrt{5}\cos t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{5}$

$$\oint_C z\, ds = \int_0^{2\pi} (5 - \sqrt{5}\cos t)\sqrt{5}\, dt = \sqrt{5}(5t - \sqrt{5}\sin t)\Big|_0^{2\pi}$$
$$= 5\sqrt{5}\cdot 2\pi = 10\sqrt{5}\pi$$

Total Surface Area $= \underline{5\pi + 5\sqrt{2}\pi + 10\sqrt{5}\pi}$ //